MATH 567: Lecture 1 (01/07/2025)

Integer and Combinatorial Optimization

↳ we will talk mostly about IP (integer programming)

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Today:
* syllabus, logistics...
* 2D LP example
* convexity, min-max as LP

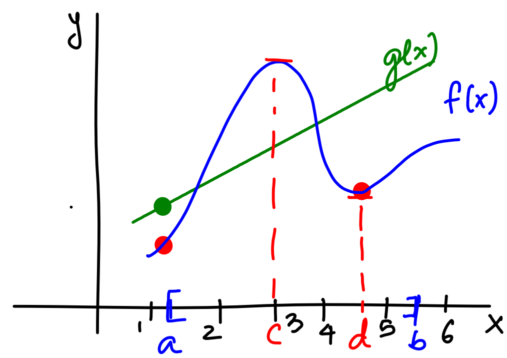
Optimization in Calculus

min $f(x)$, $x \in [a,b]$

$f'(x) = 0 \to$ critical points

$f''(x) > 0 \to$ minima

Also consider end points included in the interval.



$g(x)$: linear, min $g(x)$, $x \in [a,b]$ $\Rightarrow$ just check end points!

Linear optimization generalizes the special linear 1D case to higher dimensions.

Integer optimization in 1D: min $f(x)$, $x \in [a,b]$, $x \in \mathbb{Z}$ → set of integers

→ insist on finding integer optimal solutions

A standard linear program (LP)

$$\begin{array}{ll} \max & \bar{c}^T \bar{x} \\ \text{s.t.} & A\bar{x} \leq \bar{b} \\ & \bar{x} \in \mathbb{R}^n_{\geq 0} \end{array}$$

$\bar{c}, \bar{x} \to$ vectors (lower-case letters w/ bar)

↳ my notation in these notes!

or $=$ (as used in many books)

Here is an example of LP formulation, and graphical solution.

(Taken from *Introduction to Mathematical Programming* by Winston and Venkataramanan.)

Farmer Jones must decide how many acres of corn and wheat to plant this year. An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week. An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week. Wheat can be sold at \$4 per bushel, and corn at \$3 per bushel. Seven acres of land and 40 hours of labor per week are available. Government regulations require that at least 30 bushels of corn need to be produced in each week. Formulate and solve an LP which maximizes the total revenue that Farmer Jones makes.

Decision variables (d.v.s):

x_i = # acres of crop i, $i = 1, 2$ (1 = corn, 2 = wheat)

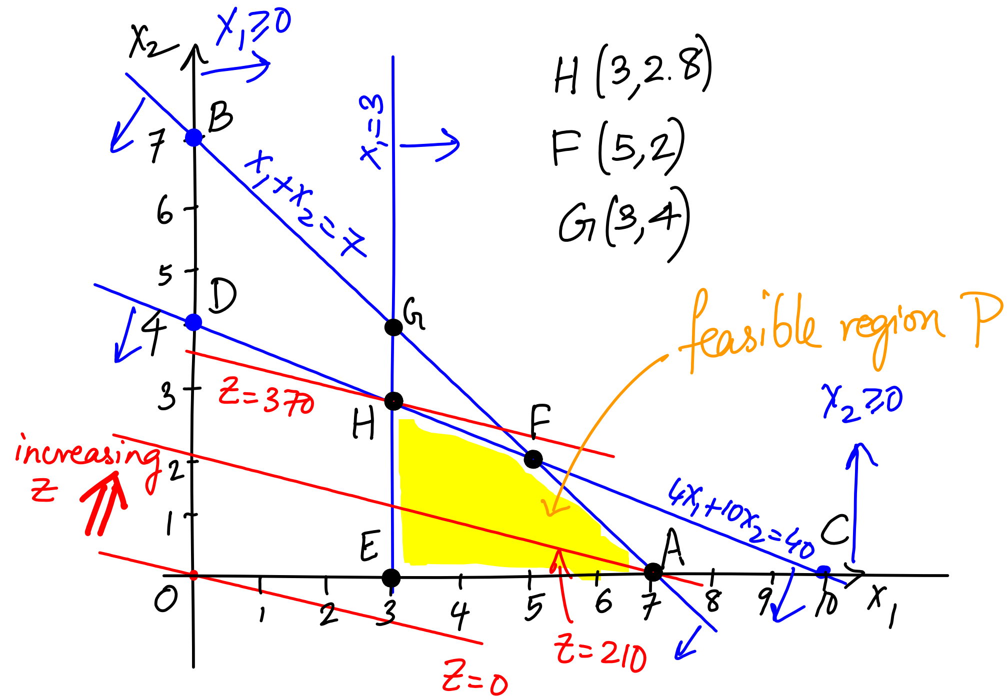
$$\left.\begin{array}{llll} \text{maximize} & z = 3 \cdot 10 \cdot x_1 + 4 \cdot 25 \cdot x_2 & & \text{(total revenue)} \\ \text{s.t.} & x_1 + x_2 & \leq 7 & \text{(total land)} \\ & 4x_1 + 10x_2 & \leq 40 & \text{(total labor hrs)} \\ & 10x_1 & \geq 30 & \text{(min corn)} \\ & x_1, x_2 & \geq 0 & \text{(non-neg)} \end{array}\right\}$$

subject to

linear program (LP)

The above model is an LP formulation. You need to specify the d.v.s, and describe the objective function and constraints as linear functions or (in)equalities in the d.v.'s.

Graphical Solution of the LP



slide z-line up (in the direction of increasing z)

Optimal solution is at H (3, 2.8), with optimal $z^* = 370$.

Points to note

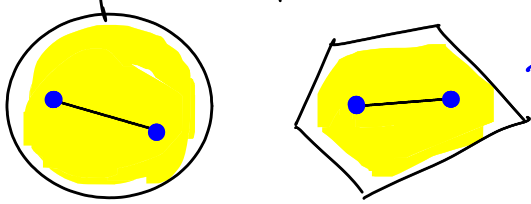
* Optimal solution (if one exists) occurs at a corner point (vertex)
* the feasible region P is convex

The first result depends on the fact that P is convex — indeed, we will strive hard to utilize convexity whenever possible!

Def P is convex if $\forall \; 0 \leq \lambda \leq 1$, $\bar{x}_1, \bar{x}_2 \in P$,
$$\lambda \bar{x}_1 + (1-\lambda) x_2 \in P.$$

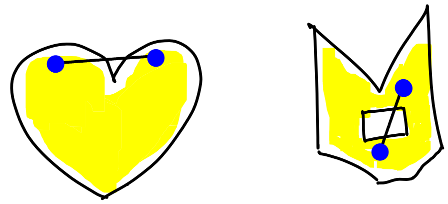
In words, the line segment connecting $\bar{x}_1$ and $\bar{x}_2$ lies in P.

Examples of convex regions:



We will talk a lot about polytopes and polyhedra

not convex!

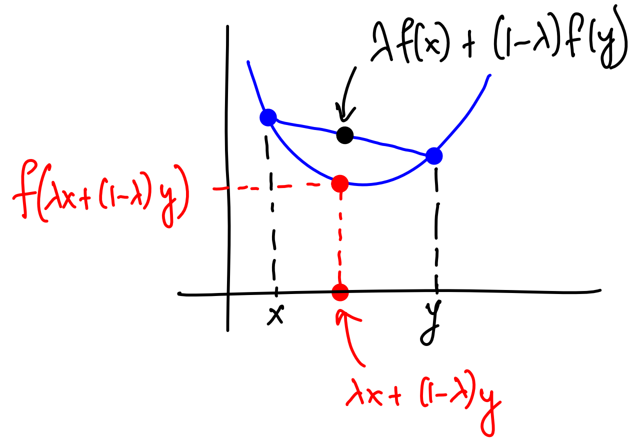


<u>**Convex function**</u> $f : \mathbb{R}^n \to \mathbb{R}$, is **convex** if

$\forall\ \bar{x}, \bar{y} \in \mathbb{R}^n$, $\lambda \in [0,1]$, we have

$$f(\lambda \bar{x} + (1-\lambda)\bar{y}) \leq \lambda f(\bar{x}) + (1-\lambda) f(\bar{y})$$

$\geq$ → concave



Again, convexity plays a critical role in how we formulate LPs and integer LPs. There are certain scenarios which could be modeled as LPs due to convexity, but other that cannot be!

Theorem: Let $f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ be convex. Then

$$f(\bar{x}) = \max_{i=1,\dots,m} f_i(\bar{x}) \text{ is convex.}$$

We will stick to the finite cases in this class, i.e., m and n in the above theorem are finite.

Proof Let $\bar{x}, \bar{y} \in \mathbb{R}^n$, $\lambda \in [0,1]$.

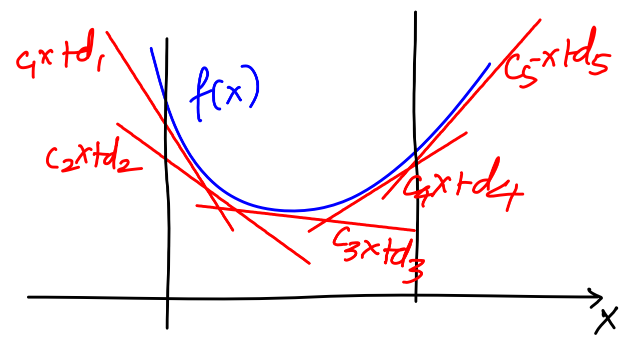
$$\begin{aligned}
f(\lambda\bar{x} + (1-\lambda)\bar{y}) &= \max_{1\le i\le m} f_i(\lambda\bar{x} + (1-\lambda)\bar{y}) \\
&\le \max_{1\le i\le m} \lambda f_i(\bar{x}) + (1-\lambda) f_i(\bar{y}) \quad \text{as each } f_i \text{ is convex} \\
&\le \lambda \underbrace{\left[\max_{1\le i\le m} f_i(\bar{x})\right]}_{f(\bar{x})} + (1-\lambda) \underbrace{\left[\max_{1\le i\le m} f_i(\bar{y})\right]}_{f(\bar{y})} \\
&= \lambda f(\bar{x}) + (1-\lambda) f(\bar{y})
\end{aligned}$$

□

Def $f(\bar{x}) = \max_{1\le i\le m} (\bar{c}_i^T \bar{x} + d_i)$, $\bar{c}_i \in \mathbb{R}^n$, $d_i \in \mathbb{R}$ is a **piecewise linear (PL) convex function**, with convexity following from the previous result.

A (general) convex function could be approximated efficiently using a PL convex function in certain optimization problems.

The more number of linear pieces we use, the finer the approximation of $f(x)$ is.



Consider the following generalization of LP:

$$\begin{aligned} \min \quad & \max_{1\le i\le m} (\bar{c}_i^T\bar{x}+d_i) \\ \text{s.t.} \quad & A\bar{x}\le\bar{b} \\ & \bar{x}\geq\bar{0} \end{aligned}$$

not an LP as written (as the objective is not $\bar{c}^T\bar{x}$)

We can write an equivalent LP with an extra variable and m extra constraints:

LP
$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & z\geq \bar{c}_i^T\bar{x}+d_i, \quad i=1,\dots,m \\ & A\bar{x}\le\bar{b} \\ & \bar{x}\geq\bar{0} \end{aligned}$$

Note: We are able to model as an LP because the objective function is a min-max one.

What if it were a min-min or a max-max one?

$$\begin{aligned} & \max\ z \\ & z\geq \bar{c}_i^T\bar{x}+d_i \\ & A\bar{x}\le\bar{b} \\ & \bar{x}\geq\bar{0} \end{aligned}$$

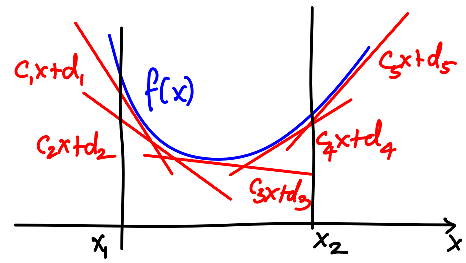
We will have to use integer variables!

this LP is unbounded!

Analogy:

z models a "blanket" that stays above all the lines (linear "pieces") $\Leftarrow$ $z \geq \bar{c}_i^T \bar{x} + d_i$, $i=1,\dots,m$.

Minimizing z pushes the blanket plush against the pieces from above, and hence z models the function as desired.



If we maximize z instead, the blanket is pulled up, and there is no limit on how far it can be pulled up. The problem is unbounded in this case!

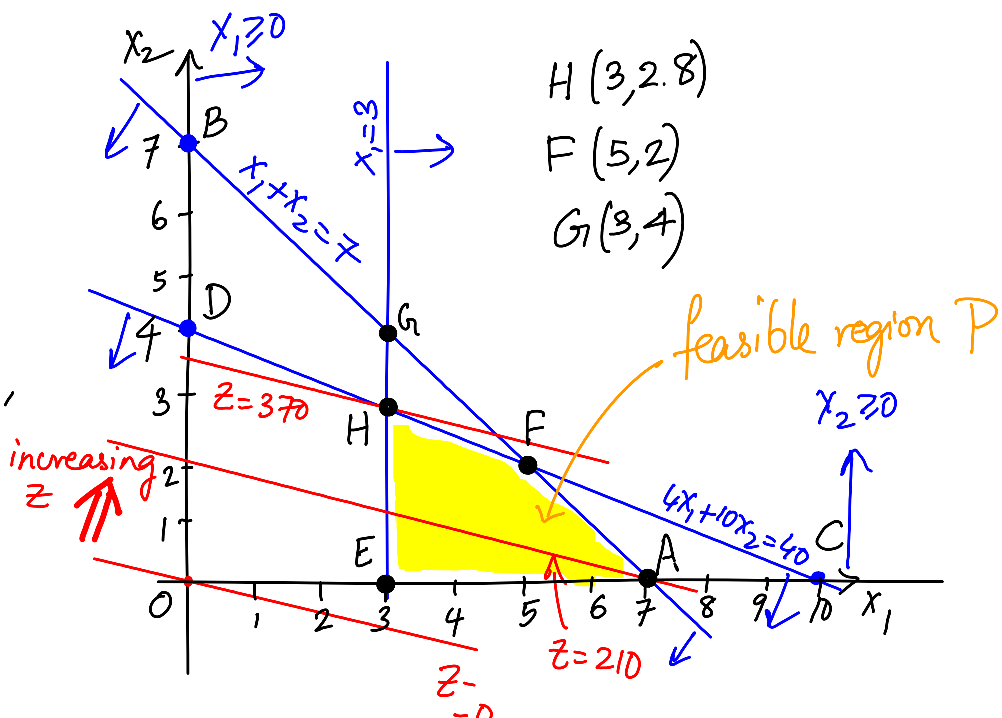
The first instinct on given an IP is to drop the integer restrictions and solve the underlying LP. May be then we can "round" the LP solution to a nearest integral solution to get a solution for the IP. But this idea fails in many cases.

Farmer Jones LP:

$H(3, 2.8)$ is the optimal solution.

If $x_1, x_2 \in \mathbb{Z}$ on top, rounding H will put you outside P!

$(3,3) \notin P$



Rounding down gives (3,2), which is feasible. But the optimal integer solution is F(5,2).