MATH 567: Lecture 20 (03/25/2025)

Today: * A different proof for $P_j(K) \subseteq Q_j(K)$
* Disjunctive programming

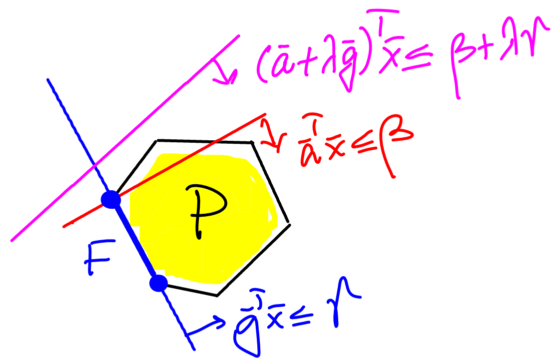
Recall $Q_j(K) = \text{conv}\left([K \cap (x_j=0)] \cup [K \cap (x_j=1)]\right)$; **Theorem 14** $P_j(K) = Q_j(K)$.

A different proof We first state and prove a lemma.

Lemma 15 Let $P = \{\bar{x} \in \mathbb{R}^n \mid A\bar{x} \leq \bar{b},\ \bar{g}^T\bar{x} \leq \gamma\}$ and
$F = P \cap \{\bar{x} \in \mathbb{R}^n \mid \bar{g}^T\bar{x} = \gamma\}$, i.e.,
F is a face of P. Suppose $\bar{a}^T\bar{x} \leq \beta$ is valid for F but not valid for P. Then there exists $\lambda \geq 0$ such that
$(\bar{a} + \lambda\bar{g})^T\bar{x} \leq \beta + \lambda\gamma$ is valid for P.

Proof (Farkas' lemma)

$$F \left\{ \begin{array}{ll} \left.\begin{array}{l} A\bar{x} \leq \bar{b} \\ \bar{g}^T\bar{x} \leq \gamma \end{array}\right\} P & \begin{array}{l} \bar{u} \geq \bar{0} \\ v_1 \geq 0 \end{array} \\ -\bar{g}^T\bar{x} \leq -\gamma & \quad v_2 \geq 0 \end{array} \right.$$



Can derive $\bar{a}^T\bar{x} \leq \beta$ from F:

$$\Rightarrow \left.\begin{array}{l} \bar{u}^TA + v_1\bar{g}^T - v_2\bar{g}^T = \bar{a}^T \\ \bar{u}^T\bar{b} + v_1\gamma - v_2\gamma \leq \beta \end{array}\right\} \quad \begin{array}{l} \bar{u}^TA + v_1\bar{g}^T = \bar{a}^T + v_2\bar{g}^T \\ \bar{u}^T\bar{b} + v_1\gamma \leq \beta + v_2\gamma \end{array}$$

Another inequality can be derived using multipliers $(\bar{u}, v_1)$ from P:

$\Rightarrow (\bar{a} + v_2\bar{g})^T\bar{x} \leq \beta + v_2\gamma$ is valid for P.

$\lambda = v_2$ works for the lemma. $\square$

Proof for $P_j(K) \subseteq Q_j(K)$

Suppose $\bar{a}^T\bar{x} \leq \beta$ is valid for both $K \cap (x_j = 0)$ and $K \cap (x_j = 1)$, two faces of K.

$-x_j \leq 0$ $x_j \leq 1$ or $-(1-x_j) \leq 0$

We use Lemma 15 to simultaneously lift this inequality so that it is valid for all of K.

$\Rightarrow$ Find $\lambda \geq 0$ and $\mu \geq 0$ such that

$$\bar{a}^T\bar{x} - \lambda x_j \leq \beta \text{ is valid for } K, \qquad (1)$$

and
$$\bar{a}^T\bar{x} - \mu(1-x_j) \leq \beta \text{ is valid for } K. \qquad (2)$$

WLOG, (1) and (2) are already part of $A\bar{x} \leq \bar{b}$. Else, we could derive them from $A\bar{x} \leq \bar{b}$ using nonnegative multipliers.

Consider the following scaled inequalities in $M_j^{NL}(K)$

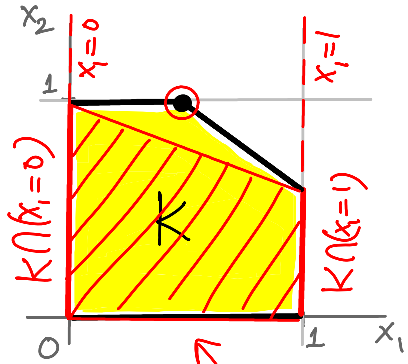
$$\left.\begin{aligned} (1-x_j)(\bar{a}^T\bar{x} - \lambda x_j) &\leq (1-x_j)\beta \\ x_j(\bar{a}^T\bar{x} - \mu(1-x_j)) &\leq x_j\beta \\ x_j(1-x_j) &= 0 \end{aligned}\right\} \text{ parts of } M_j^{NL}(K).$$

Adding them gives $\bar{a}^T\bar{x} - (\lambda+\mu)\underbrace{(\cancel{x_j}(1-x_j))}_{=0} \leq \beta$.

$\Rightarrow$ $\bar{a}^T\bar{x} \leq \beta$ is valid for $M_j^{NL}(K)$, and hence for $P_j(K)$.

$\square$

Disjunctive Programming (DP) and Disjunctive Convexification



$(x_1=0) \vee (x_1=1)$ disjunction

Disjunctive Convexification: take intersection of K with a disjunction, and take convex hull. The idea is to obtain a sharper formulation in the process.

A **disjunctive program** (DP) is an optimization problem of the following form:

$$\left\{\begin{array}{ll} \max & \bar{c}^T\bar{x} \\ \text{s.t.} & \bar{x}\in K \\ & \bar{x}\in D_1\cap D_2\cdots\cap D_p \end{array}\right\} \text{(DP)}$$

where $K=\{\bar{x}\in\mathbb{R}^n \mid A\bar{x}\le\bar{b}\}$ and

$$D_i = D_{i_1}\cup D_{i_2}\cdots\cup D_{i_{k_i}}, \quad i=1,\dots,p, \quad \text{i.e., the}$$

i^{th} disjunction has k_i alternatives.

The set K is the LP-relaxation of (DP).

D_{i_ℓ} are polyhedra $(\ell=1,\dots,k_i)$, and are called the terms in the i^{th} disjunction.

Examples

1. $k_i = 2 \ \forall i$, $D_{i_1} = \{\bar{x} \in \mathbb{R}^n \mid x_i = 0\}$ and $D_{i_2} = \{\bar{x} \in \mathbb{R}^n \mid x_i = 1\}$.
   $K = \{\bar{x} \in \mathbb{R}^n \mid A\bar{x} \le \bar{b}\}$ includes the bounds $0 \le x_i \le 1$ for $i = 1, \ldots, p$, $p \le n$.
   Then (DP) is the usual 0-1 (M)IP. if $p < n$, we get MIP.

2. $k_i = 2 \ \forall i$, $D_{i_1} = \{\bar{x} \in \mathbb{R}^n \mid x_{\ell_1} = 0\}$ and $D_{i_2} = \{\bar{x} \in \mathbb{R}^n \mid x_{\ell_2} = 0\}$,
   while K contains bounds $x_\ell \geq 0 \ \forall \ell$. Here (DP) is a linear program with complementarity constraints of the form $x_{\ell_1} x_{\ell_2} = 0$.

We can solve DP "easily" if we have all inequalities for $\text{conv}\left[K \cap (D_1 \cap \cdots \cap D_p)\right]$. But when do we get efficient representations?

Notation $A \cap_c B = \text{conv}(A \cap B)$.

Assuming we know the sharp representation of $K \cap D_i \ \forall i$, we can devise a theoretical cutting plane algorithm for DP.

Theoretical Cutting Plane Algorithm

Step 0 $K_0 = K = \{\bar{x} \mid A\bar{x} \le \bar{b}\}$.

Step i $(1 \le i \le p)$

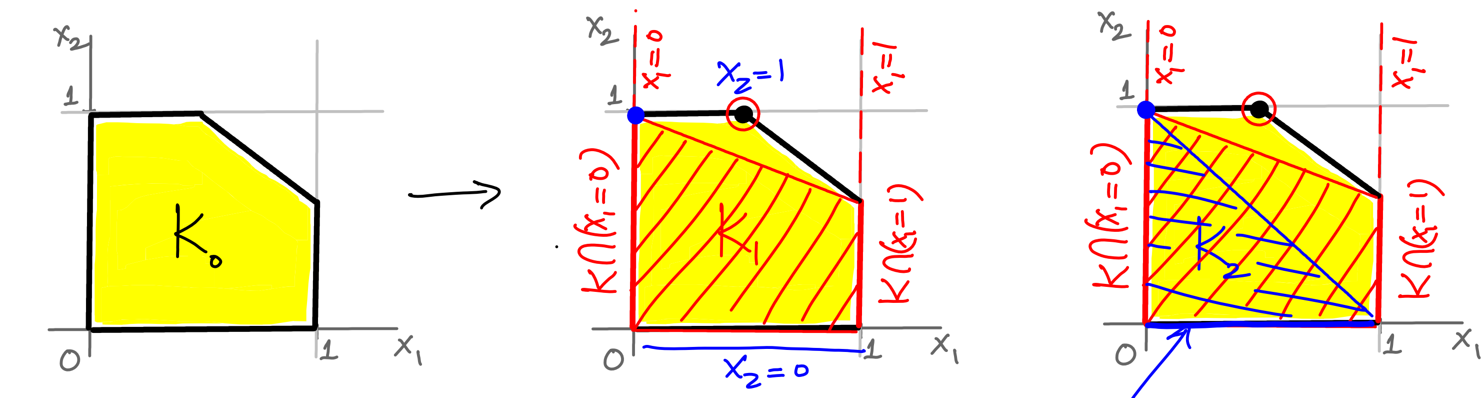
if $K_{i-1} = \{\bar{x} \mid A_{i-1}\bar{x} \le \bar{b}^{i-1}\}$

generate all inequalities for $K_{i-1} \cap_c D_i$

set $K_i = K_{i-1} \cap_c D_i = \text{conv}\left[K_{i-1} \cap D_i\right]$;

$i \leftarrow i+1$;

Example $D_1 = \{x_1 = 0 \vee x_1 = 1\}$, $D_2 = \{x_2 = 0 \vee x_2 = 1\}$.



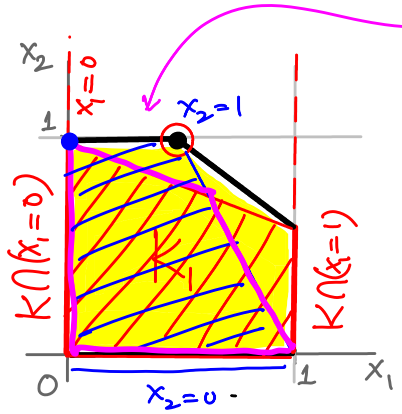
convex hull of the three vertices (0,0), (1,0) and (0,1), which is the triangle.

Here, $K_2 = \mathrm{conv}[K \cap D_1 \cap D_2] = K \cap_c (D_1 \cap D_2)$.

Hence, this is a "good instance".

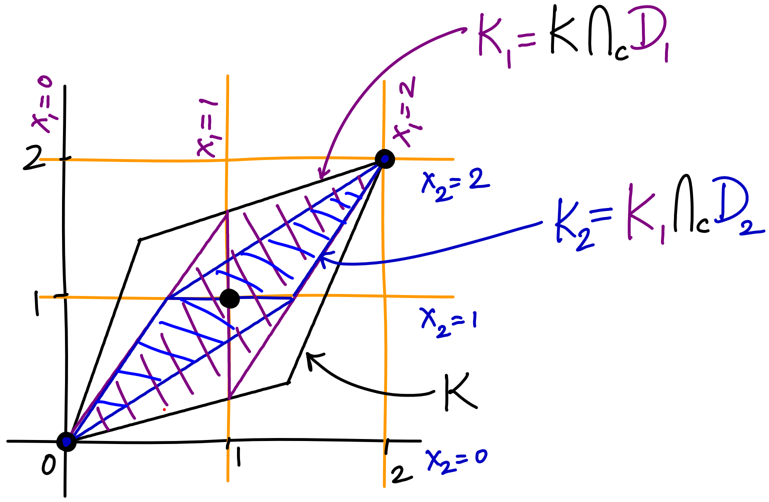
Remark $\mathrm{conv}[K \cap D_1] \cap \mathrm{conv}[K \cap D_2] \neq \mathrm{conv}[K \cap D_1 \cap D_2]$!

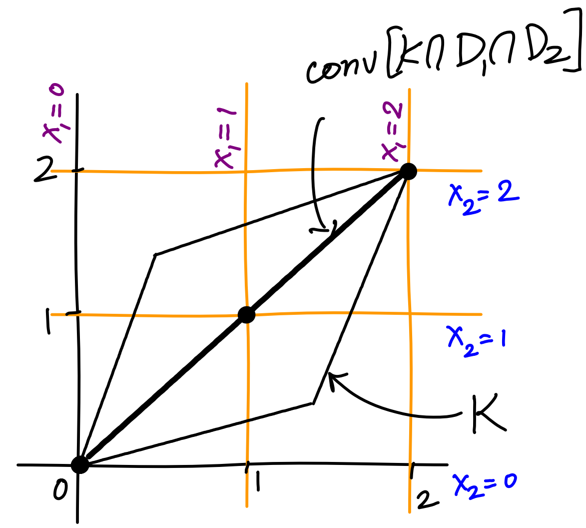
(typically)



A "bad" instance $p=2,\ k_i=3,\ i=1,2$

$$D_i = (x_i=0) \vee (x_i=1) \vee (x_i=2), \quad i=1,2.$$





Here, $K_2 \neq \text{conv}[K \cap D_1 \cap D_2]$.

Q: When is $K_p = K \cap_c (D_1 \cap \cdots \cap D_p)$, where

$$K_p = (\cdots((K \cap_c D_1) \cap_c D_2) \cdots \cap_c D_p) \ ?$$

In general '$=$' does not hold above.

Notice that in Example 1, the disjunctions $(x_1=0) \vee (x_1=1)$ and $(x_2=0) \vee (x_2=1)$ both defined faces of K, while this was not the case in Example 2 ($x_1=1$ and $x_2=1$ both did not define faces of K). It turns out that if all terms in each disjunction defines a face of K, things are nice!