MATH 567: Lecture 24 (04/08/2025)

Today: * p-center and p-median problems

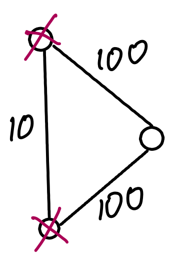
p-center and p-median problems

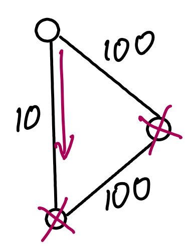
IDEA: Locate at most p facilities, assign every customer to one facility

minimize →
- total distance of all customers to their facility } p-median — warehouses
- maximum distance of any one customer to their facility } p-center — fire stations

Consider this instance:
(numbers on edges are d_{ij})

d_{ij}: distance between nodes i & j





objective function values

2-center	2-median
100	100
10	10

IP formulation

$$x_j = \begin{cases} 1 & \text{if facility located at } j, \; j \in N = \{1,2,\dots,n\} \\ 0, & \text{otherwise, and} \end{cases}$$

$$y_{ij} = \begin{cases} 1 & \text{if customer } i \text{ is assigned to facility } j, \; i,j \in N \\ 0, & \text{otherwise.} \end{cases}$$

Constraints

$$\sum_j y_{ij} = 1 \quad \forall i$$ (customer i assigned to one facility)

$$\sum_j x_j \le p$$ (at most p facilities installed)

$$\left(y_{ij} \le x_j \quad \forall i,j \right)$$ (customer i can be assigned to j only if facility located there) → disaggregated constraints

or

$$\left(\sum_{i=1}^{n} y_{ij} \le n x_j \quad \forall j \right)$$ → aggregated constraints

$$x_j, \; y_{ij} \in \{0,1\}$$

Objective functions

$$\min \quad z = \sum_j \sum_i d_{ij} y_{ij}$$ (p-median)

d_{ij} = distance between nodes i and j

$$\left.\begin{array}{l} \min \quad z \\ z \ge d_{ij} y_{ij} \quad \forall i,j \end{array}\right\}$$ (p-center)

To also minimize # poles ($\le p$) among optimal solutions to p-center/p-median, we can set

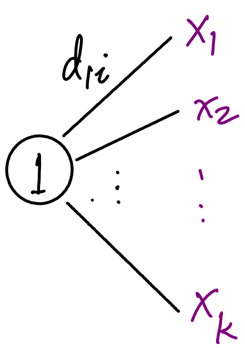
$$\min \quad z + \varepsilon \sum_{j=1}^{n} x_j \qquad \text{where} \quad 0 < \varepsilon < \min_{i,j} \{d_{ij}\} / n$$

Applications p-median : warehousing → minimize the total distance from warehouses to all retailers.

p-center : firestations → minimize the largest distance of any customers from their assigned fire station.

Claim It is enough to put $0 \le y_{ij} \le 1$, i.e., $y_{ij} \in \{0,1\}$ is not needed.

Proof Suppose $y_{11} + y_{12} + \cdots + y_{1k} = 1$ and $y_{1j} = 0$ for $j > k$, where $0 < y_{1i} < 1$ for $i = 1, \ldots, k$. So, customer 1 is partially assigned to facilities $1, \ldots, k$.



Then we can make $y_{1,i^*} = 1$ where

$$d_{1,i^*} = \min_{1 \le i \le k} \{d_{1i}\}, \text{ and set } y_{1i} = 0 \text{ for } i \ne i^*.$$

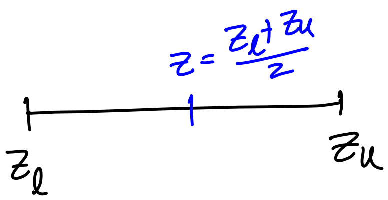
Hence, there always exists an optimal solution with $y_{ij} \in \{0,1\}$. $\square$

x_j's still need to be set as binary variables.

We will look at heuristics for the p-center and p-median problems. The MIP formulations are often too big to handle, even for moderately sized problems.

Binary Search Heuristic for p-center

Let $z_\ell = 0$, $z_u = M$ (lower, upper bounds on z, the objective function)



while $(|z_u - z_\ell| > \varepsilon)$ → this ε is not related to the ε we used to scale secondary objective function (two pages ago)!

{

$$z = \frac{z_\ell + z_u}{2};$$

$$C_i = \{ j \mid d_{ij} \le z \}$$

Solve a set covering problem (SCP) with C_i's;
↳ use greedy/modified greedy

if optimal value (SCP) $> p$

set $z_\ell = z$; → ignores lower half of $[z_\ell, z_u]$; as we need to increase the allowed distances

else

set $z_u = z$; → ignore upper half of $[z_\ell, z_u]$; We are able to satisfy requirements with $\le p$ facilities, so we could try to tighten the distances now.

end

} (end_while)

Greedy heuristic for the p-median problem

If $\boxed{p=1,}$ we can locate the one facility optimally.

Let $z_j = \sum_{i=1}^{n} d_{ij}$, pick j with minimal z_j.

$\boxed{(p-1) \longrightarrow p:}$ Assuming the first $(p-1)$ facilities stay, we locate the p^{th} facility optimally.

Let X_{p-1} be the set of $(p-1)$ facilities already located.

We compute

$$z_j = \sum_i \text{dist}(i, X_{p-1} \cup \{j\}) \quad \text{where}$$

$$\text{dist}(i, X_{p-1} \cup \{j\}) = \min_{k \in X_{p-1} \cup \{j\}} \{d_{ik}\}, \quad \text{and}$$

Select j with the smallest z_j, set $X_p = X_{p-1} \cup \{j\}$.

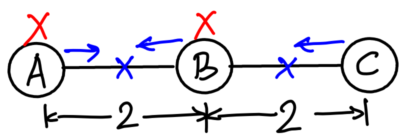
In other words, we are choosing the p^{th} facility greedily.

We could do a clean-up type run through the selected facilities after each facility is picked, or after, say, every 10^{th} facility is picked.

Absolute p-center problem

Here, we can locate facilities on the edges of the graph, in addition to locating on its nodes.

Note: The original version is called the vertex p-center problem.



Objective function values (z):

p	vertex	absolute
1	2	2
2	2	1

How do we solve the absolute p-center problem? It appears that there could be infinitely many new candidate facility locations!

Even though there might be infinitely many possibilities for the absolute p-center problem, we can show the following result.

We can select finitely many points, $\mathcal{N}$, on the edges and then

Absolute p-center problem on $G = (V, E)$ $\equiv$ vertex p-center on $G' = (V \cup \mathcal{N}, E')$, where

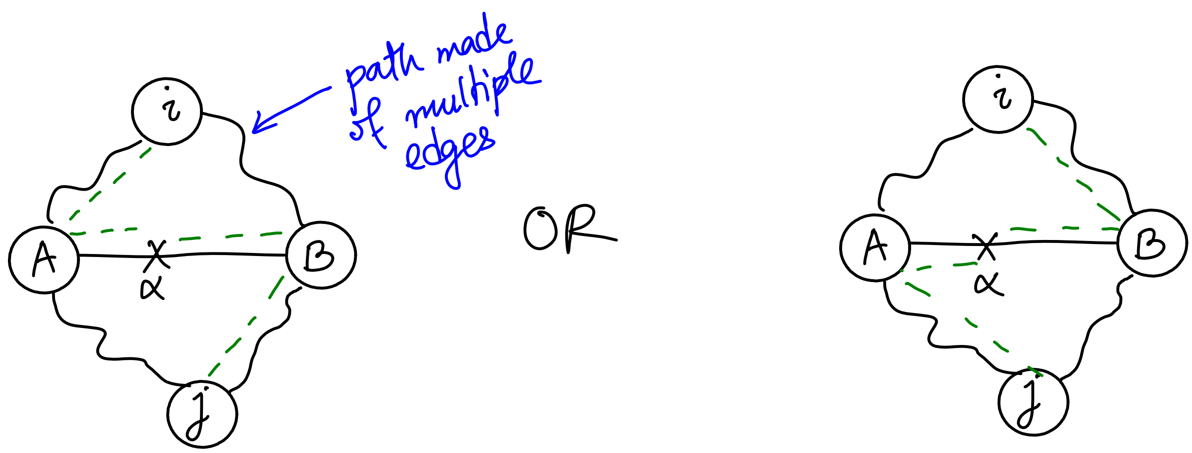
E' consists of original edges in E split into two or more edges when vertices from $\mathcal{N}$ are added.

Def A facility α is a **local center** for nodes i and j if

$$d(i,\alpha) = d(j,\alpha) \leq d(k,\alpha) \;\forall\; k \neq i, j, A, B, \alpha.$$

where α is located on $\overleftrightarrow{AB}$, and i, j, k are assigned to α.

The shortest paths from α to i and j must go alternatively through A and B, as shown below.



Claim There exists an optimal solution to the absolute p-center problem where every facility is a local center for some i and j.