MATH 567: Lecture 25 (04/10/2025)

Today: * Absolute p-center problem
* DKP, lattices, and basis reduction

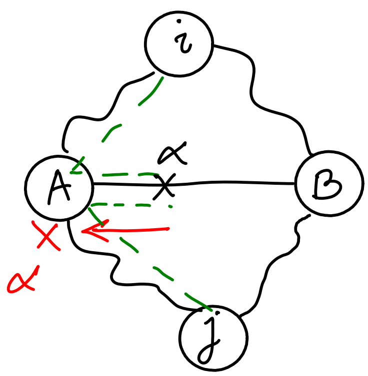
Recall..

Claim There exists an optimal solution to the absolute p-center problem where every facility is a local center for some i and j.

Proof Assume a facility α is not a local center.

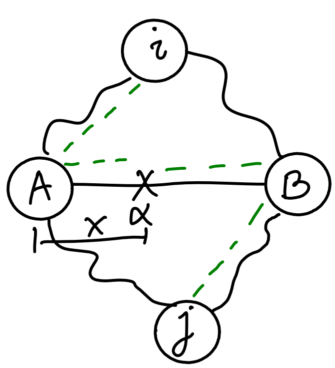
$\Rightarrow d(i,\alpha) > d(k,\alpha)\ \forall k \neq i$, and k, i assigned to α.

We can move α closer to i until $d(i,\alpha) = d(k,\alpha)$ for some k. We still keep i the farthest node from α. This sliding will not worsen the objective function. $\square$



Suppose α is a local center for i, j, but the shortest paths from i & j to α both go through A, say. Here, we could slide α along AB to A, and still maintain α being a local center for i, j, while maintaining $d(i,\alpha) = d(j,\alpha)$.

Hence, we should have the following set-up (or its complementary one):



With $d(A,\alpha) = x$, we must have

$$d(i,\alpha) = d(i,A) + x = d(j,\alpha) \leq d(j,A) + x$$

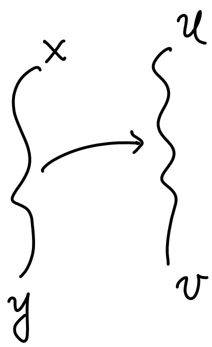
$$\Rightarrow d(i,A) \leq d(j,A).$$

Similarly, we need

$$d(i,B) \geq d(j,B) \text{ here.}$$

Notation

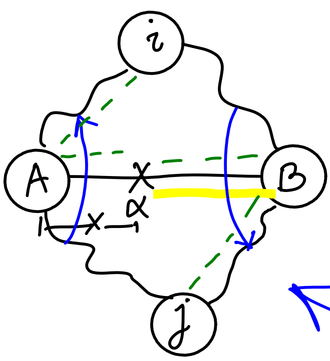
$d(x,y) \geq d(u,v)$



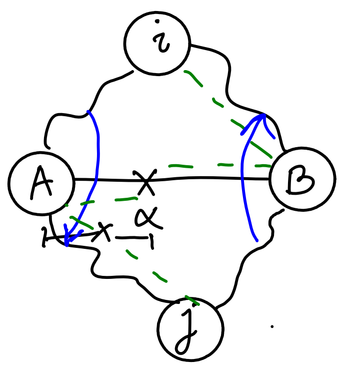
$\equiv$ path x-y is greater or equal in length than path u-v.

Conditions for existence of a local center for i,j on $\overline{AB}$

Condition 1: A local center α can exist on $\overline{AB}$ for i,j if



OR



holds.

with the first option

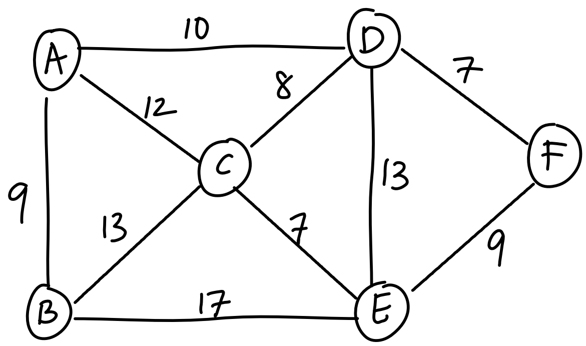
Condition 2 If condition 1 holds, the location of the local center is given by

$$d(i,A) + x = \big(d(A,B) - x\big) + d(j,B)$$

$$\Rightarrow \quad x = \frac{d(A,B) + d(j,B) - d(i,A)}{2}.$$

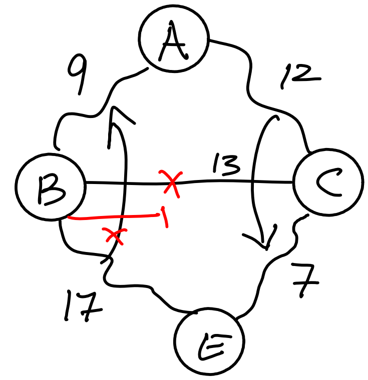
We can derive a similar expression for the second option.

Example



Identify (possible) local centers:

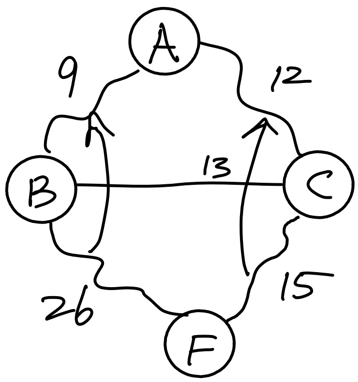
1. On BC



$$9 + x = 7 + (13 - x)$$

$$\Rightarrow x = \frac{11}{2} = 5.5.$$

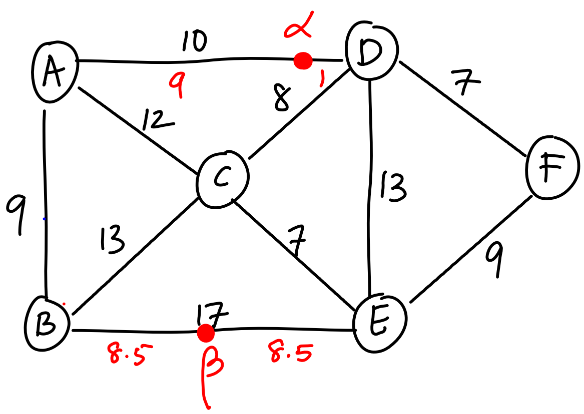
2. Center on BC for $\{i,j\} = \{A,F\}$?



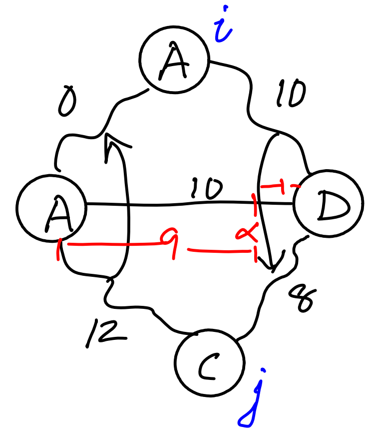
Condition 1 does not hold!

We can check and identify all possible local centers in this manner.

An Optimal solution:



Assignments: $\begin{cases} \alpha : A, C, D, F \\ \beta : B, E \end{cases}$



Optimal z value $= 9$ here.

(for (A,α) and (C,α)).

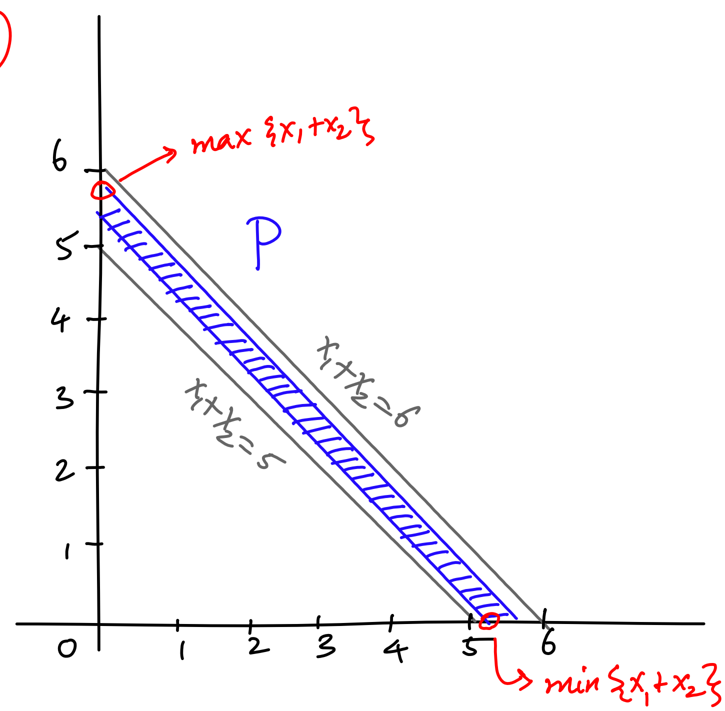
Lattice basis reduction and IP

Basis reduction (BR) is an important method used in proving several theoretical results in IP, e.g., devising polynomial time algorithms to solve IP when the dimension is considered fixed.

We will introduce the basic concepts related to lattices and BR, and discuss how these concepts help in solving certain classes of IPs that are otherwise hard for the usual methods involving branch-and-cut.

We start by considering a 2D knapsack feasibility problem (KP).

$$\text{(KP)}\begin{cases} P\begin{cases} 106 \le 21x_1 + 19x_2 \le 113 \\ 0 \le x_1, x_2 \le 6 \end{cases} \\ x_1, x_2 \in \mathbb{Z} \end{cases}$$



Qn: Is $P \cap \mathbb{Z}^2 = \phi$?

Yes, but B&B will take $\approx (6)^2$ nodes to prove this fact when binary branching on x_1, x_2 is used.

But consider the direction $x_1 + x_2$.

$$\max\{x_1+x_2 \mid \bar{x} \in P\} = 5.94 \ \left(\tfrac{113}{19}\right), \text{ and}$$

$$\min\{x_1+x_2 \mid \bar{x} \in P\} = 5.04 \ \left(\tfrac{106}{21}\right).$$

Hence, branching on $x_1 + x_2$ proves infeasibility at the root node.

This 2D example is an instance of a more general knapsack problem of the form

$$(P)\left\{\begin{array}{c}\beta' \le \bar{a}^T\bar{x} \le \beta \\ \bar{0} \le \bar{x} \le \bar{u}\end{array}\right\} \\ \bar{x} \in \mathbb{Z}^n \quad (KP)$$

where $P \cap \mathbb{Z}^n = \phi$ by design, i.e., (KP) is integer-infeasible, but B&B branching on the individual variables takes an exponential (in n) number of BB nodes to prove it.

In particular, we study a class of (KP) problems where $\bar{a} = \bar{p}M + \bar{r}$, with $\bar{p} \geq \bar{0}$, $M >> 1$, and $\bar{p}, \bar{r} \in \mathbb{Z}^n$, $M \in \mathbb{Z}$. Since $\bar{a}$ decomposes in this fashion, we call these problems decomposable knapsack problems (DKPs).

The bounds $\beta' \le \beta$ are also $\in \mathbb{Z}_{>0}$, and are chosen such that $P \cap \mathbb{Z}^n = \phi$, i.e., DKP is integer-infeasible.

In the example, $\bar{p} = \begin{bmatrix}1\\1\end{bmatrix}$, $M = 20$, $\bar{r} = \begin{bmatrix}1\\-1\end{bmatrix}$, $\bar{a} = \begin{bmatrix}21\\19\end{bmatrix}$, $\beta' = 106$, $\beta = 113$.

Further, if we consider $\max\{\bar{p}^T\bar{x} \mid \bar{x} \in P\}$ and $\min\{\bar{p}^T\bar{x} \mid \bar{x} \in P\}$, we get values of the form $(k+1) - \delta$ and $k + \delta'$ where $0 < \delta, \delta' < 1$. for some $k \in \mathbb{Z}$ Hence branching on $\bar{p}^T\bar{x}$ proves infeasibility at the root node.

How do we locate the "good" direction $\bar{p}$ (we do not know beforehand that $\bar{a}$ decomposes in this fashion)?

Lattices

The lattice spanned by $[\bar{b}_1, \ldots, \bar{b}_n]$, $\bar{b}_i \in \mathbb{R}^m$, $i = 1, \ldots, n$, is the set of integer linear combinations of $\bar{b}_i$'s.

$$\mathcal{L}([\bar{b}_1, \ldots, \bar{b}_n]) = \left\{ \sum_{j=1}^{n} \bar{b}_j x_j \;\middle|\; x_j \in \mathbb{Z} \;\forall j \right\}.$$

With $B = [\bar{b}_1 \ldots \bar{b}_n]$, we can write

$$\mathcal{L}(B) = \left\{ B\bar{x} \mid \bar{x} \in \mathbb{Z}^n \right\}. \qquad B \in \mathbb{R}^{m \times n}$$

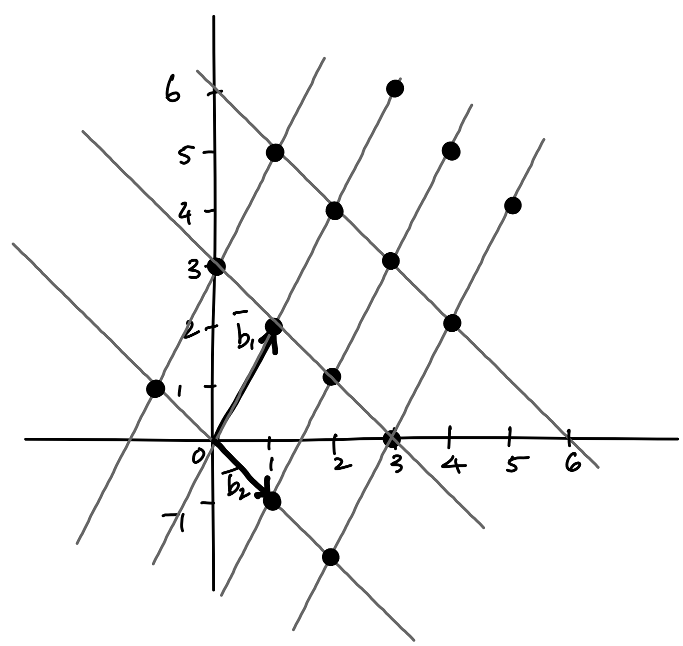
Note: $\text{span}(B) = \{ B\bar{y} \mid \bar{y} \in \mathbb{R}^n \}$ is the linear subspace spanned by the columns of B.

One could think of lattices as the main spaces in "integer linear algebra".

e.g., $\bar{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\bar{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$B = [\bar{b}_1\ \bar{b}_2]$

$\mathcal{L}(B)$ consists of the grid points shown here as $\bullet$.

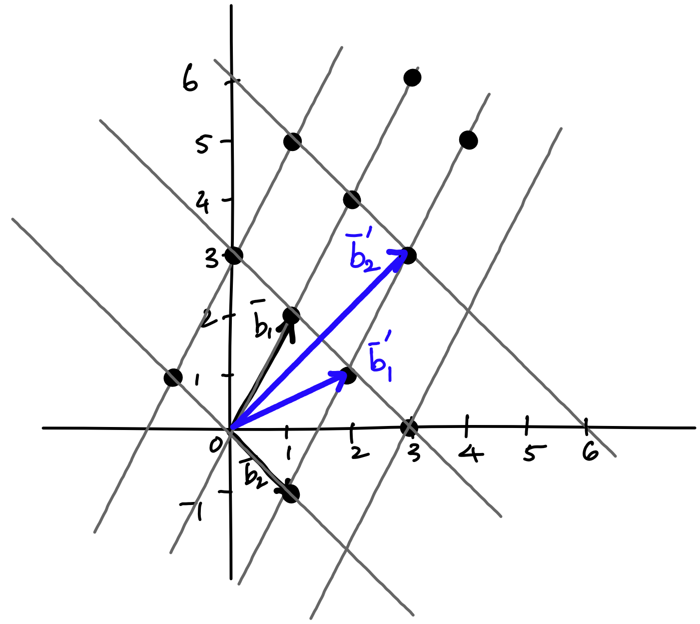


With $\bar{b}_1' = \bar{b}_1 + \bar{b}_2 = \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\bar{b}_2' = 2\bar{b}_1 + \bar{b}_2 = \begin{bmatrix}3\\3\end{bmatrix}$, we get another basis $B' = [\bar{b}_1' \ \bar{b}_2']$ for $\mathcal{L}(B)$.

In the example, we had $\mathcal{L}(B) = \mathcal{L}(B')$

with $B = [\bar{b}_1 \ \bar{b}_2] = \begin{bmatrix}1 & 1\\2 & -1\end{bmatrix}$ and

$B' = [\bar{b}_1' \ \bar{b}_2'] = \begin{bmatrix}2 & 3\\1 & 3\end{bmatrix}$.



Note: $B' = BU$ where $U = \begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}$.

$$\underbrace{\begin{bmatrix}2 & 3\\1 & 3\end{bmatrix}}_{B'} = \underbrace{\begin{bmatrix}1 & 1\\2 & -1\end{bmatrix}}_{B}\underbrace{\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}}_{U}.$$ Here

$\det(U) = -1$.

In general, given two bases B, B' for a lattice $\mathcal{L}$, we have $B' = BU$ for unimodular U, i.e., $\det(U) = \pm 1$.

Any vector in $\mathcal{L}(B)$ is $\bar{v} = B\bar{x}$, $\bar{x} \in \mathbb{Z}^2$. We can write

$$\bar{v} = B\bar{x} = BUU^{-1}\bar{x} = B'\bar{x}'.$$

Notice that U^{-1} has integer entries, as $\det U = \pm 1$.

Also, $\bar{x}$ and $\bar{x}'$ are unique for a given $\bar{v}$.