MATH 567 : Lecture 3 (01/16/2025)

Today: * More MIP formulations
* modeling tools for BIPs

Recall : min-max objective functions and constraints — could be modeled as linear programs.

e.g., $\min\{|x|\} \longrightarrow \min\{\max\{x,-x\}\}$

$\longrightarrow \min\{z \mid z \geqslant x,\ z \geqslant -x\}.$

Similarly, we could model $\max\{\cdot,\cdot\cdot\} \leq b$ or $\min\{\cdot\cdot,\cdot\cdot\} \geqslant b$ constraints as equivalent linear systems. For instance,

$$|x| \leq 5 \longrightarrow \max\{x,-x\} \leq 5 \longrightarrow x \leq 5,\ -x \leq 5.$$

But $|x| \geqslant 4$ cannot be modeled as an LP. In particular,

~~$x \geqslant 4$ and $-x \geqslant 4$~~ is not what we want.

Will have to use an extra binary variable to model which of two options holds in this case.

Recall : Fixed charge :

$$\begin{aligned} \min\ & f_1 y_1 + \cdots & (f_1 > 0) \\ \text{s.t.}\ & \cdots \\ & x_1 \leq M_1 y_1 & y_1 \in \{0,1\} \end{aligned}$$

We will see another problem class where fixed charge shows up. Later, we will see how to force the relation between x_1 and y_1 without relying on the $\min f_1 y_1$ objective function.

5. Uncapacitated lot sizing (ULS)

* 1 product, n time periods $(t = 1, \dots, n)$
* $d_1, \dots, d_n$: demand in each time period
* $f_1, \dots, f_n$: fixed cost for making any > 0 # items in each time period
* $c_1, \dots, c_n$: unit production cost in each time period
* $h_1, \dots, h_n$: unit holding (or storage) costs (h_t : cost for storing one unit from period t to $t+1$)

Goal: production plan that minimizes total cost.

Assumptions: * infinite production capacity (no storage capacity as well)
* no units to start with, or at end

d.v.s:

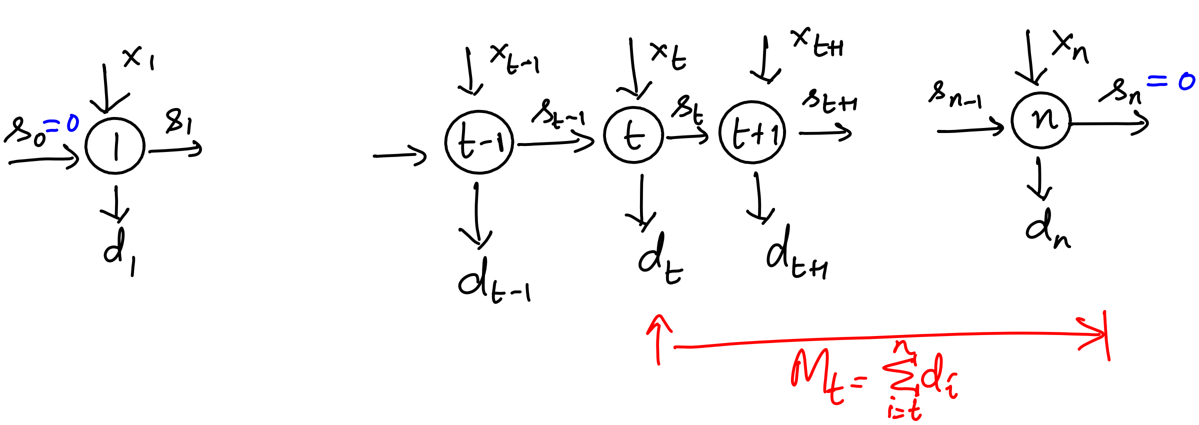
x_t = # units produced in period t, $t = 1, \dots, n$ (≥ 0, continuous)

s_t = # units stored from period t to $t+1$, $t = 0, \dots, n$ (≥ 0, continuous)

$$y_t = \begin{cases} 1 & \text{if } x_t > 0 \\ 0 & \text{o.w.} \end{cases}, \quad t = 1, \dots, n$$

to capture the fixed charge terms

Here is a schematic:



d_t's are data given to us

Here is the MIP: → we do have an MIP, as s_t, x_t are continuous, while y_t is binary

$$\min \sum_{t=1}^{n} f_t y_t + \sum_{t=1}^{n} c_t x_t + \sum_{t=1}^{n} h_t s_t \qquad \text{(total cost)}$$

$$\text{s.t.} \qquad s_0 = 0, \quad s_n = 0 \qquad \text{(no start/end inventory)}$$

$$\underbrace{s_{t-1} + x_t}_{\text{inflow}} = \underbrace{d_t + s_t}_{\text{outflow}}, \quad t = 1, \ldots, n \qquad \text{(flow balance)}$$

$$x_t \leq M_t y_t \quad t = 1, \ldots, n \qquad \text{(forcing constraints)}$$

$$s_t \geq 0, \; x_t \geq 0, \; y_t \in \{0,1\} \;\; \forall t \qquad \text{(var. restrictions).}$$

What should M_t be? Any large enough (>0) number will work, But ideally, use the smallest M_t that works.

$$M_t = \sum_{i=t}^{n} d_i \quad \text{will work here.}$$

We will spend a lot of time on details such as the choice of M_t, and how they affect the "strength" of the formulation.

If we allow backlogging, demand in period t could be satisfied by (part of) x_j for $j > t$. In this case,

$$M_t = \sum_{i=1}^{n} d_i \quad \text{will work,}$$

since all the demand could potentially be satisfied by producing in the same single period.

6. (General) Piecewise linear function (not convex in the interesting case)

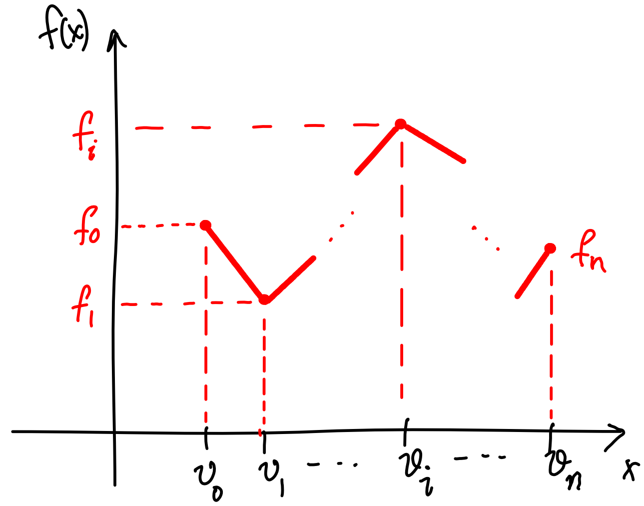
x : scalar

$$f(x) = \begin{cases} f_i, & \text{if } x = v_i \quad (i = 0, \dots, n) \\ \text{linear}, & \text{if } v_i \le x \le v_{i+1} \quad (i = 0, \dots, n-1) \end{cases}$$

If $x = \lambda_i v_i + \lambda_{i+1} v_{i+1}$

$\lambda_i, \lambda_{i+1} \ge 0$, $\lambda_i + \lambda_{i+1} = 1$, then

$f(x) = \lambda_i f_i + \lambda_{i+1} f_{i+1}$



Let $\delta_i = v_i - v_{i-1}$. We let

$$s_i = \frac{f_i - f_{i-1}}{\delta_i}, \quad i = 1, \dots, n \text{ (slopes, can be } \ge 0 \text{ or } \le 0\text{)}.$$

Let x_i be "the portion of x in $[v_{i-1}, v_i]$", $i = 1, \dots, n$.

If we

1. write $x = v_0 + \sum_{i=1}^{n} x_i$

$$g = f_0 + \sum_{i=1}^{n} s_i x_i$$

$$0 \le x_i \le \delta_i,$$

2. somehow enforce

"if $x_{i+1} > 0$ then $x_i \ge \delta_i$", for $i = 1, \dots, n-1$

3. plug in g for $f(x)$;

then we're done!

2. Equivalent logical expression:

$A \Rightarrow B \equiv$
$\neg A$ or B
"not"

"either $x_{i+1} \le 0$ or $x_i \ge \delta_i$"

$$-x_i + \delta_i \le 0$$

Let y_i and z_i are 0-1 variables

$$\boxed{\begin{aligned} x_{i+1} &\le \delta_{i+1} y_i \\ -x_i + \delta_i &\le \delta_i z_i \\ y_i + z_i &= 1 \\ y_i, z_i &\in \{0,1\} \end{aligned}}$$

if $x_{i+1} > 0$ then $y_i = 1$
$\Rightarrow z_i = 0$

$\forall i = 1, \dots, n-1$

assuming XOR "exclusive OR"
A or B, but not both

If $x_{i+1} > 0$, then $y_i = 1 \Rightarrow z_i = 0$ (as $y_i + z_i = 1$).
$\Rightarrow -x_i + \delta_i \le 0 \Rightarrow x_i \ge \delta_i$

Can simplify:

$$x_{i+1} \le \delta_{i+1} y_i$$

$$-x_i + \delta_i \le \delta_i (1 - y_i)$$

as $y_i + z_i = 1$

$$x_i \ge \delta_i y_i$$

i.e., $\delta_i y_i \le x_i \le \delta_i y_{i-1}$, $i = 1, \dots, n-1$

$x_{i+1} > 0 \Rightarrow y_i = 1 \Rightarrow y_{i-1} = 1, y_{i-2} = 1, \dots, y_1 = 1.$

So, we can force both implications for $y_i = \begin{cases} 1, & \text{if } x_i > 0 \\ 0 & \text{o.w} \end{cases}$ using constraints, i.e., do not have to rely on a min $f_i y_i$ objective function.

We present one last formulation instance...

7. Semicontinuous variable

Need that "x does not take values that are too small".

e.g., if you buy any of a stock option, you need to buy at least 100 of them.

statement: x is zero or is at least ℓ (>0) (and $\leq M$)

Model: $\boxed{\ell y \leq x \leq M y, \; y \in \{0,1\}}$

We now consider some themes/governing principles for writing all such formulations.

1. Modeling with only 0-1 variables

$x_1, x_2, \ldots$ are 0-1 (binary) variables

Notation

$L_i \equiv (x_i = 1)$

$\vee \equiv$ OR, $\wedge \equiv$ AND

$\Rightarrow \equiv$ "implies", $\Leftrightarrow \equiv$ "equivalent"

$\neg \equiv$ NOT (negation)

These are standard notation used in mathematical logic. We will start with statements, and then try to write the model, i.e., set of inequalities, that represents the statement.

Examples

statement	model (constraints)
1. $L_1 \vee L_2 \vee \cdots \vee L_n$	$x_1 + x_2 + \cdots + x_n \geq 1$
2. $L_1 \Rightarrow L_2$	$x_1 \leq x_2$
3. $L_1 \Leftrightarrow (L_2 \wedge L_3)$	

i.e.,

$$\left\{\begin{array}{l} L_1 \Rightarrow (L_2 \wedge L_3) \\ L_1 \Leftarrow (L_2 \wedge L_3) \end{array}\right.$$

$L_1 \Rightarrow (L_2 \wedge L_3)$ → $x_1 \leq x_2$, $x_1 \leq x_3$

$L_1 \Leftarrow (L_2 \wedge L_3)$ → think about it! We'll present the model in the next lecture...