MATH 567: Lecture 4 (01/21/2025)

Today: * conjunctive normal form (CNF)
* model with 0-1 and continuous variables
* arbitrary disjunctions

Modeling with 0-1 variables (continued...)

3. $L_1 \Leftrightarrow (L_2 \wedge L_3)$

i.e.,

$$\left\{\begin{array}{l} L_1 \Rightarrow (L_2 \wedge L_3) \\ L_1 \Leftarrow (L_2 \wedge L_3) \end{array}\right.$$

$L_1 \Rightarrow (L_2 \wedge L_3) \rightarrow x_1 \le x_2,\ x_1 \le x_3$ OR $2x_1 \le x_2 + x_3$

$L_1 \Leftarrow (L_2 \wedge L_3) \rightarrow \cancel{x_1 \ge \frac{x_2+x_3}{2}}$ will force $x_1 = 1$ when $x_2=1, x_3=0$!

$\cancel{x_1 \ge x_2 x_3}$ nonlinear!

$\cancel{2x_1 = x_2 + x_3}$ $x_2 = x_3 = 0$ forces $x_1 = 0$!

$x_1 \ge x_2 + x_3 - 1$

Q. Is there a general method to model any logical statement?

YES! As long as the statement is in a "nice" form.
And every statement has such a "nice" form!

Def A **literal** is an elementary statement, e.g., L_i, $\neg L_j$.

A **clause** is a set of literals connected with "OR" ($\vee$)
e.g., $L_1 \vee L_3$, $\neg L_2 \vee L_4 \vee \neg L_5$.

Def A logical statement is in **conjunctive normal form** (CNF) if it is a set of clauses connected by ANDs ($\wedge$).

e.g., $(L_1 \vee L_3) \wedge (\neg L_2 \vee L_3 \vee \neg L_5) \wedge (\neg L_3 \vee L_7)$

is in CNF.

If a statement is in CNF, it is easy to write down its representative model using inequalities.

e.g., $\left\{\begin{array}{r} x_1 + x_3 \geq 1 \\ (1-x_2) + x_3 + (1-x_5) \geq 1 \\ (1-x_3) + x_7 \geq 1 \end{array}\right\}$ is a model for the statement in CNF above.

Claim Every (finite) statement involving $\vee, \wedge, \neg, \Rightarrow, \Leftrightarrow$ has a CNF. The CNF may not be unique.

Some Rules for doing transformations

① $L_1 \wedge (L_2 \vee L_3) \equiv (L_1 \wedge L_2) \vee (L_1 \wedge L_3)$

$\cdot \quad +$

② $L_1 \vee (L_2 \wedge L_3) \equiv (L_1 \vee L_2) \wedge (L_1 \vee L_3)$

$\cdot \quad +$

③ $\neg (L_1 \wedge L_2) \equiv \neg L_1 \vee \neg L_2$

④ $\neg (L_1 \vee L_2) \equiv \neg L_1 \wedge \neg L_2$

⑤ $L_1 \Rightarrow L_2 \overset{\text{def}}{\equiv} \neg L_1 \vee L_2$

We could replace literals with clauses, or more general statements in the above rules, and they still hold, e.g., $C_1 \Rightarrow C_2 \equiv \neg C_1 \vee C_2$.

clauses

Examples

1. $(L_2 \wedge \cdots \wedge L_n) \Rightarrow L_1 \equiv \neg(L_2 \wedge \cdots \wedge L_n) \vee L_1$

$$\equiv (\neg L_2 \vee \neg L_3 \vee \cdots \vee \neg L_n) \vee L_1$$

$$\equiv \neg L_2 \vee \neg L_3 \vee \cdots \vee \neg L_n \vee L_1$$

which is in CNF.

model: $(1-x_2)+(1-x_3)+\cdots+(1-x_n)+x_1 \geq 1$

2. $(L_1 \wedge L_2) \vee (L_3 \wedge (L_4 \vee L_5))$

$$\equiv ((L_1 \wedge L_2) \vee L_3) \wedge ((L_1 \wedge L_2) \vee (L_4 \vee L_5))$$

$$\equiv ((L_1 \vee L_3) \wedge (L_2 \vee L_3)) \wedge [(L_1 \vee (L_4 \vee L_5)) \wedge (L_2 \vee (L_4 \vee L_5))]$$

$$\equiv (L_1 \vee L_3) \wedge (L_2 \vee L_3) \wedge (L_1 \vee L_4 \vee L_5) \wedge (L_2 \vee L_4 \vee L_5)$$

which is in CNF.

model: $\left\{ \begin{array}{l} x_1 + x_3 \geq 1 \\ x_2 + x_3 \geq 1 \\ x_1 + x_4 + x_5 \geq 1 \\ x_2 + x_4 + x_5 \geq 1 \end{array} \right\}$

2. Modeling with 0-1 and continuous variables

Let $y \in \{0,1\}$, $\bar{x} \in \mathbb{R}^n$

Statement : $y=1 \Rightarrow A\bar{x} \leq \bar{b}$

Assume $\exists\, \bar{u} \geq \bar{0}$: $A\bar{x} \leq \bar{b} + \bar{u}$ is always true.

Then $\boxed{A\bar{x} \leq \bar{b} + \bar{u}(1-y)}$ is the model.

3. Modeling arbitrary disjunctions

$\bar{x} \in \mathbb{R}^n$

$$(A_1\bar{x} \leq \bar{b}^1) \vee (A_2\bar{x} \leq \bar{b}^2) \vee \cdots \vee (A_k\bar{x} \leq \bar{b}^k) \quad (*)$$

Assume $\{\bar{x} \mid A_i\bar{x} \leq \bar{b}^i\} \neq \emptyset$. if one system is $\emptyset$, then we could remove it from $(*)$

In words, $(*)$ says "$\bar{x}$ satisfies one of the k systems."

Note that some of the statements using literals L_i would fit this framework. At the same time, this is a much more general statement. We'll consider two approaches to model this statement. The first one looks quite similar to the previous case of $y=1 \Rightarrow A\bar{x} \leq \bar{b}$.

big-M representation

Assumption 1 $\exists\ \bar{u}^i \geq \bar{0}$ such that $\forall \bar{x}$ that satisfy $A_j\bar{x} \leq \bar{b}^j$ for some j, $A_i\bar{x} \leq \bar{b}^i + \bar{u}^i$ holds $\forall i$.

Let $y_i \in \{0,1\}$, $i = 1, \ldots, k$. → models whether the i^{th} disjunction holds

$$\boxed{\begin{aligned} & A_i\bar{x} \leq \bar{b}^i + \bar{u}^i(1-y_i), \quad i=1,\ldots,k \\ & y_1 + y_2 + \cdots + y_k \geq 1 \\ & y_i \in \{0,1\}, \quad i=1,\ldots,k. \end{aligned}} \quad \text{———} \quad (\cancel{x}\text{-big}M)$$

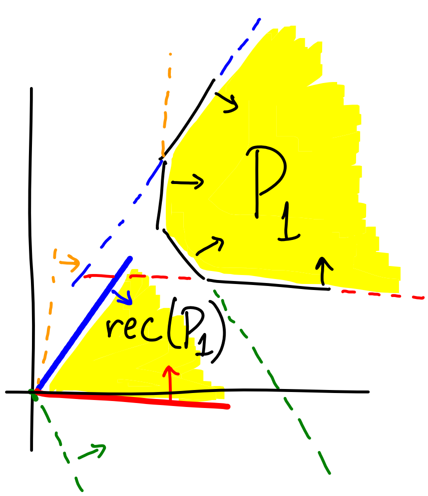
Sharp formulation

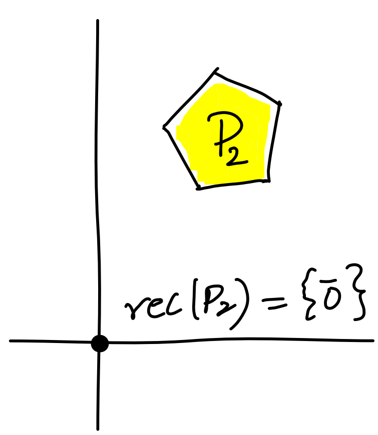
Assumption 2 $\exists\ C$ such that

$C = \{\bar{x} \mid A_i\bar{x} \leq \bar{0}\}$, $i = 1, \ldots, k$ is independent of i.

Def The **recession cone** of polyhedron $P = \{\bar{x} \mid A\bar{x} \leq \bar{b}\}$ is

$$\mathbf{rec}(P) = \{\bar{x} \mid A\bar{x} \leq \bar{0}\}.$$





If P is a polytope, i.e., a closed polyhedron, then $\text{rec}(P) = \{\bar{0}\}$, the origin.

$$\boxed{\begin{aligned} A_1\bar{x}^1 &\le \bar{b}^1 y_1 \\ &\vdots \\ A_k\bar{x}^k &\le \bar{b}^k y_k \\ \bar{x}^1+\bar{x}^2+\cdots+\bar{x}^k &= \bar{x} \\ y_1+y_2+\cdots+y_k &= 1 \\ y_i &\in \{0,1\} \end{aligned}}$$

($\ast$-sharp)

sharp, as exactly one (out of k) options is forced to hold

We now prove the correctness of ($\ast$-sharp).

Theorem 1 $\bar{x}$ satisfies $\ast$ $\iff$ $\exists\,(\bar{x}^1,\ldots,\bar{x}^k, y_1,\ldots,y_k)$ such that $(\bar{x}, \bar{x}^1,\ldots,\bar{x}^k, y_1,\ldots,y_k)$ satisfies ($\ast$-sharp).

Proof ($\Rightarrow$) $\bar{x}$ satisfies $\ast$.

WLOG, let $A_1\bar{x} \le \bar{b}^1$. We can choose

$$\left.\begin{aligned} y_1=1,\ y_2=\cdots=y_k=0 \\ \bar{x}^1=\bar{x},\ \bar{x}^2=\cdots=\bar{x}^k=\bar{0} \end{aligned}\right\}\ \text{satisfies } (\ast\text{-sharp}).$$

($\Leftarrow$) : in the next lecture...