MATH 567: Lecture 7 (01/30/2025)

Today: * sharp formulations
* TSP Formulations

Example $[(x_1=1) \Rightarrow \bigwedge_{j=2}^{n} (x_j=1),\ x_i \in \{0,1\} \forall i]$, P_1, P_2, (continued...)

We saw that $(\frac{1}{n-1}, 1, 0, 0 \ldots, 0) \in P_2$ (aggregated formulation). In fact, this point is a corner point of P_2, defined by the n LI constraints

$$(n-1)x_1 \leq x_2 + \cdots + x_n \quad \text{--- } (1)$$
$$x_2 \leq 1 \quad \text{--- } (2)$$
$$x_j \geq 0, j=3,\ldots,n \quad \text{--- } (n-2)$$

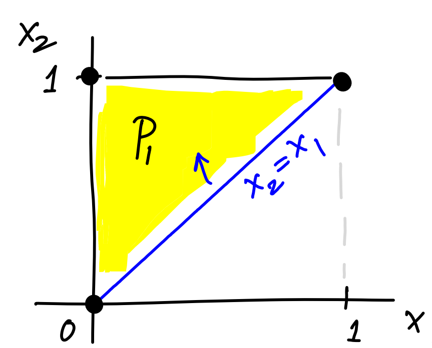
satisfied as equations. Hence P_2 is not sharp for S.

To show P_1 (disaggregated formulation) is sharp for S, we show each corner point of P_1 is integral. The inequalities defining P_1 can be broken down into 3 groups:

$$x_i \geq x_1, \quad i=2,\ldots,n \quad \text{--- } (1)$$
$$x_i \geq 0, \quad i=1,\ldots,n \quad \text{--- } (2)$$
$$x_i \leq 1, \quad i=1,\ldots,n \quad \text{--- } (3)$$

First, check intuition in 2D:

$$P_1 = \{\bar{x} \in \mathbb{R}^2 \mid x_2 \geq x_1,\ 0 \leq x_i \leq 1,\ i=1,2\}.$$



Indeed, all three corner points are integral!

In general, we consider a few cases:

(i) All $(n-1)$ inequalities from (1), and one from (2) or (3):

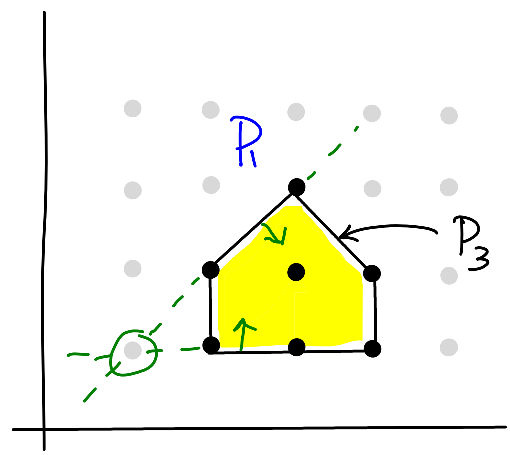
$$\Rightarrow \quad x_i = 0 \ \forall i \quad \text{or} \quad x_i = 1 \ \forall i.$$

(ii) All n inequalities from (2) or (3) : $\Rightarrow$ trivial.

(iii) $2 \leq j < n$ inequalities from (1) and $n-j$ inequalities from (2) or (3) $\Rightarrow$ Can show $x_i \in \{0,1\} \ \forall i$.

Recall that we need to consider equality versions of the constraints and their intersections to enumerate all potential corner points.

For instance, the two LI lines corresponding to two constraints indicated by green dashed lines here meet at a point that is not feasible, and hence is not a corner point of P_3.



If we can provide a sharp formulation with a "small", i.e., a polynomial number, of inequalities, then we can solve any linear optimization problem over S "easily", i.e., in polynomial time.

But for many problems, e.g., the traveling salesman problem (TSP), the sharp formulation has exponentially many inequalities.

We now consider several examples of sharp formulation.

More Examples of Sharp Formulations

1. Given a bipartite graph $G = (V_1 \cup V_2, E)$, let $S_M \subseteq \mathbb{Z}^{|E|}$ be the collection of incidence vectors of all matchings.

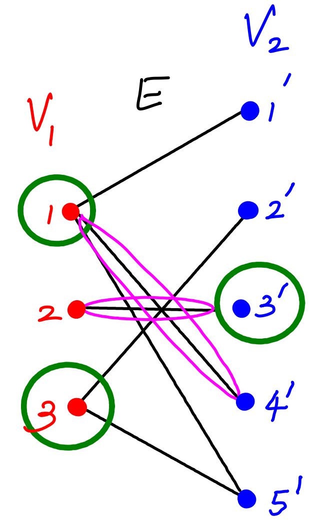
M for matching

Then $P_m = \{ \bar{x} \in \mathbb{R}^{|E|} \mid \bar{x} \geq \bar{0},$

$$\sum_{e \ni i} x_e \leq 1 \;\; \forall i \in V_1,$$

$$\sum_{e \ni j} x_e \leq 1 \;\; \forall j \in V_2 \}$$

is a sharp formulation for S_m.



$\{(1,4'), (2,3')\}$ is a matching, and $\{1,3,3'\}$ is a node cover.

2. Given $G = (V_1 \cup V_2, E)$ as above, let $S_N \subseteq \mathbb{Z}^{|V|}$ be the collection of incidence vectors of all node covers, i.e., subsets of nodes that cover all edges. Then

$V = V_1 \cup V_2$

$$P_N = \{ \bar{x} \in \mathbb{R}^{|V|} \mid 0 \leq \bar{x} \leq \bar{1}, \sum_{i \in e} x_i \geq 1 \;\; \forall e \in E \}$$

$x_i + x_j \geq 1 \;\; \forall (i,j) \in E$

is a sharp formulation for S_N.

The default problems ask to identify the maximum (cardinality) matching, and the minimum (cardinality) node cover.

3. Define S_{ij} and P_{ij} as follows, using $x_1, x_2, x_3 \in \{0,1\}$.

$$S_{ij} = \{\bar{x} \mid \bar{x} \in \{0,1\}^3, \ (x_i = 0) \vee (x_j = 0)\} \text{ and}$$

$$P_{ij} = \{\bar{x} \mid \bar{x} \in \mathbb{R}^3, \ 0 \le \bar{x} \le \bar{1}, \ \ x_i + x_j \le 1\}, \text{ for } i,j = 1,2,3, \ i \ne j.$$

We can show P_{ij} is a sharp formulation for S_{ij}.

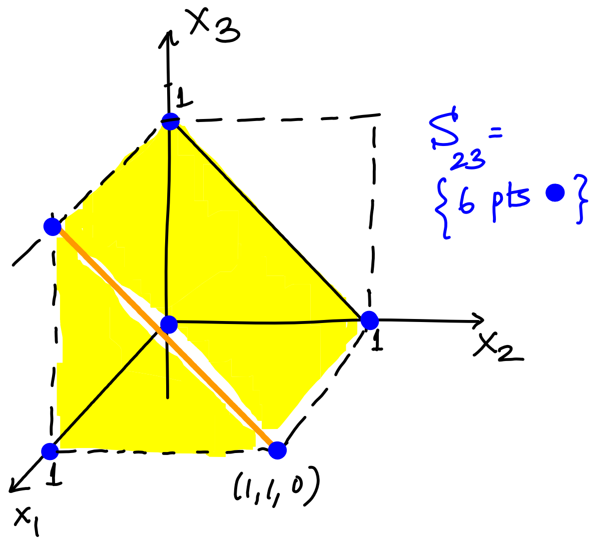
For instance,

$$S_{23} = \{(0,0,0), (1,0,0), (0,1,0), (1,1,0), (0,0,1), (1,0,1)\} \text{ and}$$

$$P_{23} = \left\{ \bar{x} \in \mathbb{R}^3 \;\middle|\; x_2 + x_3 \le 1, \ \begin{array}{l} x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0 \\ x_1 \le 1, \ \underbrace{x_2 \le 1, \ x_3 \le 1}_{\text{could remove!}} \end{array} \right\}.$$

Could check all subsets of 3 LI inequalities $\left(\le \binom{7}{3} = 35 \text{ choices}\right)$.

We could also just plot P_{23}!



In another formulation, we could drop $x_2 \le 1$ and $x_3 \le 1$, since we have $x_2 + x_3 \le 1$, which implies $x_2 \le 1$ and $x_3 \le 1$ along with $x_2 \geq 0$ and $x_3 \geq 0$.

Now, let

$$S = S_{12} \cap S_{23} \cap S_{31} = \{ \bar{x} \in \{0,1\}^3 \mid (x_1=0) \vee (x_2=0) \wedge (x_2=0) \vee (x_3=0) \wedge (x_3=0) \vee (x_1=0) \}$$

and

$$P = P_{12} \cap P_{23} \cap P_{31} = \{ \bar{x} \in \mathbb{R}^3 \mid 0 \le \bar{x} \le \bar{1},\ x_1+x_2 \le 1,\ x_2+x_3 \le 1,\ x_3+x_1 \le 1 \}.$$

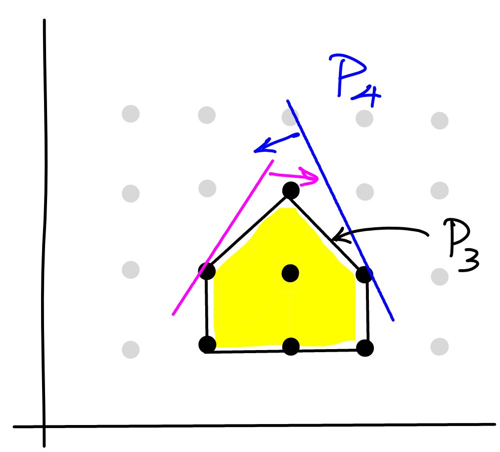
Is P a sharp formulation for S? No!

For instance, $\max \{ x_1+x_2+x_3 \mid \bar{x} \in P \}$ has a unique optimal solution at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \notin S$.

Notice that in S, no point can have two x_j's set to 1, as $(x_i=0) \vee (x_j=0)$ is true for all three pairs. So we cannot get $x_1+x_2+x_3 = 2$. But $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in P$, and indeed gives a higher value for $x_1+x_2+x_3$.

Also, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is a corner point of P: it is the point of intersection of $x_1+x_2=1$, $x_2+x_3=1$, $x_3+x_1=1$, which are LI.

There was question about whether the sharp formulation P is unique. As a set, it captures the convex hull of S, and $\text{conv}(S)$ itself is unique. But there could be alternative descriptions of P, e.g., by adding redundant constraints, as shown here with the case of P_4, which adds two redundant constraints to P_3.



Traveling Salesman Problem (TSP)

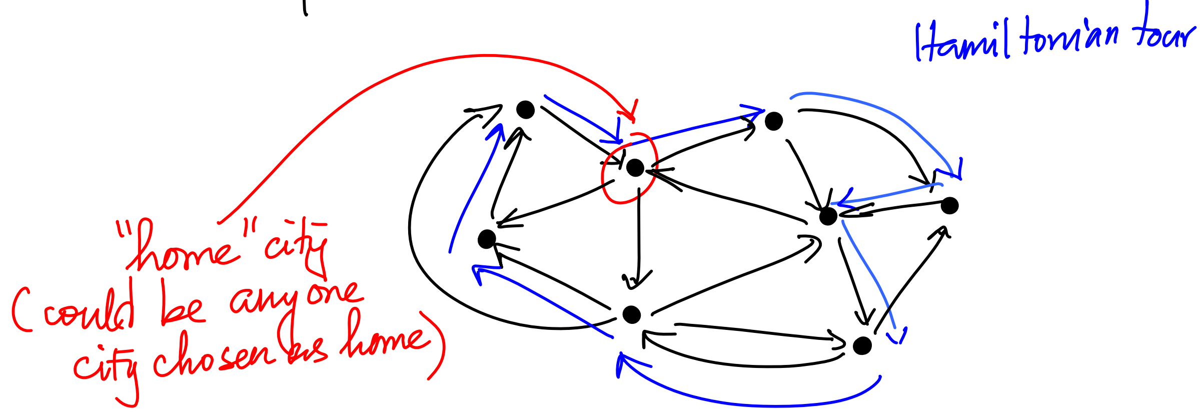
* n cities
* c_{ij} : cost (or distance) from city i to city j is defined on a directed graph $G=(V,E)$.
  ↳ directed

Goal: find a shortest (smallest total costs) Hamiltonian tour, i.e., a single directed cycle that contains all n nodes (cities), and each node is visited exactly once.

TSP is perhaps the most widely studied combinatorial optimization problem. We will consider a few different formulations for the TSP.

Forget c_{ij}'s for now.

Goal: Formulations for $S \subseteq \mathbb{Z}^{|E|}$, the set of incidence vectors of all Hamiltonian tours



$S = \{ \bar{x} \mid \bar{x}$ is the incidence vector of a Hamiltonian tour $\}$.

$x_{ij} = 1$ if $(i,j) \in$ tour

$\bar{x} \in S \Rightarrow$

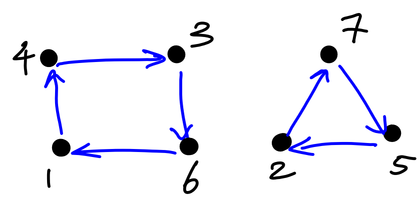
$$\left.\begin{aligned} \sum_{j:(i,j)\in E} x_{ij} &= 1 \quad \forall i \\ \sum_{j:(i,j)\in E} x_{ji} &= 1 \quad \forall i \\ 0 \le x_{ij} \le 1, \quad & \bar{x} \in \mathbb{Z}^{|E|} \end{aligned}\right\} \text{———} (1)$$

remove to get formulation, i.e., the polytope

Assume $x_{ii} = 0 \ \forall i$.

But (1) is not enough, as it allows subtours.

Here is a collection of two subtours, which together satisfy (1):



We have to avoid subtours. We examine a few different ways to avoid them. One option involves adding some extra variables, and extra constraints. The other option involves adding extra constraints using the original variables (x_{ij}).

We have to avoid subtours!

<u>First approach</u> We add u_i, $i=1,\dots,n$, node variables.

$u_i \equiv$ position of node i in tour.

any node could be the "home city"

We assume node 1 is the "home city". $u_1 = 1$.

$u_3 = 5 \Rightarrow$ Node 3 is the 5^{th} node in the tour, starting from node 1.