MATH 567 : Lecture 8 (02/04/2025)

Today: TSP formulations and comparisons

Recall $u_i \equiv$ position of node i in tour.

We want to impose

if $x_{ij}=1$ then $u_j \geqslant u_i+1$ for $i\neq 1, j\neq 1$.

We write

$$u_i - u_j + 1 \leq n(1-x_{ij}), \ \forall i\neq 1, \forall j \neq 1 \quad \text{———} \quad (2)$$

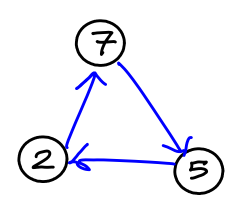
Let's check:

if $x_{ij}=1$ (2) $\Rightarrow$ $u_i - u_j + 1 \leq 0 \implies u_j \geqslant u_i+1$. ✓

if $x_{ij}=0$ (2) $\Rightarrow$ $u_i - u_j \leq n-1$. ✓

Notice that u_j need not represent the position of node j in the tour exactly. But u_j will be at least u_i+1 when $x_{ij}=1$. Thus, we could have $u_j = u_i + 5$, for instance. But even such values eliminate subtours, as they will not allow split (sub)tours as illustrated previously.

$$\{u_7 \geqslant u_2+1,\ u_2 \geqslant u_5+1,\ u_5 \geqslant u_7+1\}$$



↳ cannot hold together!

But if we add $2 \leq u_j \leq n$, $\forall j \neq 1$, we get u_j representing the position of node i exactly.

Claim $S = \{\bar{x} \in \mathbb{Z}^{|E|} \mid \exists \bar{u} : (\bar{x}, \bar{u}) \text{ satisfies } (1) \text{ and } (2)\}$.

Proof '$\subseteq$': If $\bar{x}$ is a tour, take u_i = position of node i in $\bar{x}$.

If $x_{ij} = 1$ $(2) \Rightarrow u_j \geq u_i + 1$. ✓

$x_{ij} = 0$ $(2) \Rightarrow u_i - u_j + 1 \leq n$. ✓

'$\supseteq$': $\bar{x} \notin S \overset{\text{want to show}}{\Longrightarrow}$ $\bar{x}$ violates (1) or

$\bar{x}$ satisfies (1), but there is no $\bar{u}$ to satisfy (2).

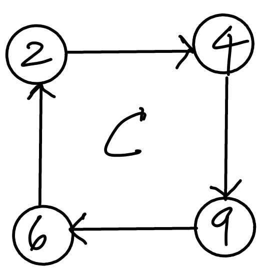
Case 1: $\bar{x}$ violates (1): trivial.

Case 2: $\bar{x}$ satisfies (1), but is not a tour.

Let C be a subtour with $1 \notin C$. In more detail,

$C = \{\overbrace{\{i_1, i_2, \ldots, i_k\}}^{\text{nodes}}$ along with edges (i_r, i_{r+1}), $r = 1, \ldots, k-1$ and (i_k, i_1), where $i_r \neq 1.\}$

e.g.,



Consider

$$u_i - u_j + 1 \leq n(1 - x_{ij}) \quad \text{— (2)}$$

for each $i, j \in C$.

Add (2) around $C \Rightarrow |C| \leq n\,(|C| - X(C))$ where

$X(C) = \sum'_{(i,j) \in C} x_{ij}$. Hence $X(C) \leq \left(1 - \frac{1}{n}\right)|C|$.

But x_{ij}'s violate this inequality!

there will be exactly $|C|$ x_{ij}'s set to 1!

$\square$

Remark If we use (1), and instead of (2), write

$$\left.\begin{array}{ll} 1 \le u_i \le n & \text{(a)} \\ u_i - u_j + 1 \le n(1 - x_{ij}), \ \forall i, \ \forall j \ne 1 & \text{(b)} \\ n - u_i \le (n-1)(1 - x_{i1}), \ \forall i \ne 1 & \text{(c)} \end{array}\right\} \quad (3)$$

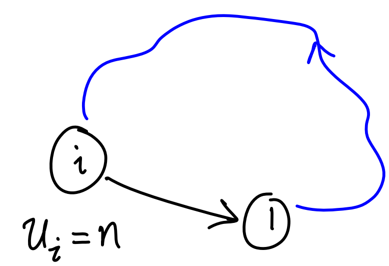
then (1) & (3) together give a valid formulation for S.

3 (b) forces $x_{ij} = 1 \Rightarrow u_j \ge u_i + 1, \ \forall i, \ \forall j \ne 1$.

3 (c) forces $x_{i1} = 1 \Rightarrow u_i \ge n \quad \forall i \ne 1$.

forces $u_i = n$, with 3(a)

u_i for node i in the are $(i, 1)$ coming into node 1 is forced to n, making i the last node in the tour.



$$S = \left\{ \bar{x} \in \mathbb{Z}^{|E|} \ \middle| \ \exists \bar{u} : (\bar{x}, \bar{u}) \text{ satisfy (1) \& (3)} \right\}.$$

(1)+(2) and (1)+(3) are quite similar to each other in terms of strength, as well as in computation.

(1)+(2) is the **Miller-Tucker-Zemlin (MTZ)** formulation.

Subtour Formulation

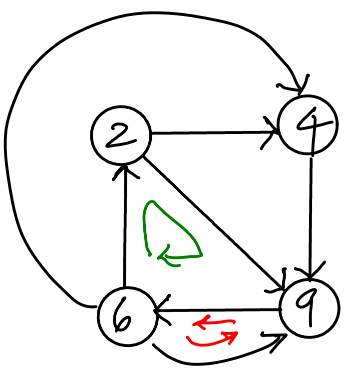
$$\boxed{\forall\, W \subseteq V,\ |W|>1,\ \sum_{\substack{i,j \in W \\ (i,j)\in E}} x_{ij} \le |W|-1} \quad \text{——— (4)}$$

more on this point in a bit...

(4) has exponentially many constraints in $|V|=n$.

(1)+(4) is a valid formulation for S, i.e.,

$$S = \{\bar{x} \in \mathbb{Z}^{|E|} \mid \bar{x} \text{ satisfies (1) and (4)}\}.$$



$W = \{2,4,6,9\}$

$$\underbrace{\sum_{\substack{i,j\in W \\ (i,j)\in E}} x_{ij}}_{7\ x_{ij}\ \text{terms here}} \le 3$$

This constraint will avoid **all** possible subtours of length 4 in G which use $\{2,4,6,9\}$, and not just the obvious one, i.e., 2-4-9-6-2.

At the same time, this constraint will allow subtours of length 2 or 3 in W, e.g., 6-9-6 or 2-9-6-2. We need the subtour constraints for $W' = \{6,9\}$ and $W'' = \{2,6,9\}$ to eliminate them.

Now, let's consider the $W \underset{?}{\subsetneq} V$ question...

Q. Should we write the subtour constraint for $W = V$? Wouldn't that eliminate all possible Hamiltonian tours?

The answers are YES and YES, as it does not matter much when considering formulations for S'. The default option is that we write the subtour constraints for all $W \subset V$, i.e., with $|W| \leq n-1$. In this case, we will indeed capture the Hamiltonian tours.

On the other hand, we could write the subtour constraint for $W = V$, in which case the Hamiltonian tours are avoided. But Hamiltonian paths are still permitted, and we could add the last missing arc in any Hamiltonian path to get the corresponding tour.

But once we involve the costs c_{ij}, we should ideally not write the subtour constraint for $W = V$. The last connecting arc (to complete the tour) could have a huge cost, affecting the minimality computations.

Also, notice that (4) is valid for $|W| = 1$, since we assume that there are no self loops, i.e., no arcs (i,i). Equivalently, $x_{ii} = 0 \; \forall i$.

Comparing MTZ and Subtour formulations

First guess

If C is a subtour (cycle) with $1 \notin C$, adding (2) around C got us

$$X(C) \leq \left(1-\frac{1}{n}\right)|C| \quad \text{———} \quad (\Delta)$$

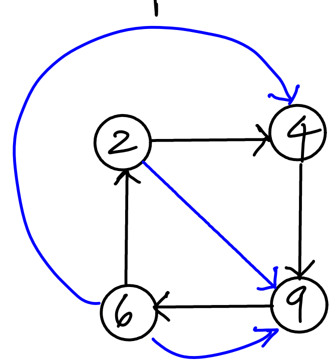
If $n=100$, $X(C) \leq 0.99|C|$, which is not very effective. Notice that $X(C) \leq |C|$ holds trivially (and from (1)). So, as n becomes larger and larger, the right-hand side value becomes closer and closer to $|C|$, while still remaining strictly smaller than $|C|$.

In the subtour formulation, using $W = C$, we get

$$\underbrace{\sum_{\substack{i,j \in W \\ (i,j)\in E}}}_{\text{more } x_{ij} \text{ terms than included in } X(C).} x_{ij} \leq \underbrace{|C|-1}_{\text{almost "1 better than } (\Delta)\text{".}} \quad \text{———} \quad (*)$$

Thus $(*)$ is stronger than (Δ) in both the left-hand and the right-hand sides. But we now make this comparison more formal.

not just the 4 arcs in C (2-4-9-6-2)



We consider

$$P_{MTZ} = \{(\bar{x},\bar{u}) \in \mathbb{R}^{|E|} \times \mathbb{R}^{|V|} \mid (\bar{x},\bar{u}) \text{ satisfy (1) and (2)}\}, \text{ and}$$

$$P_{subtour} = \{\bar{x} \in \mathbb{R}^{|E|} \mid \bar{x} \text{ satisfies (1) and (4)}\}.$$

To compare, we compute $\mathrm{Proj}_{\bar{x}}(P_{MTZ})$.

Theorem 6 $\mathrm{Proj}_{\bar{x}}(P_{MTZ}) = \{\bar{x} \mid \exists \bar{u} : (\bar{x},\bar{u}) \text{ satisfies (1) \& (2)}\}$

$$= \{\bar{x} \mid \bar{x} \text{ satisfies } (\triangle) \text{ for all cycles } C,\ 1 \notin C\}.$$

there could be exponentially many such cycles.

Indeed, P_{MTZ} is described by a small (polynomial in m,n) number of constraints using the n extra variables u_i. But an exponential number of constraints are needed to describe $\mathrm{Proj}_{\bar{x}}(P_{MTZ})$.

Proof P_{MTZ} is given by (1) and

$$u_i - u_j + 1 \le n(1 - x_{ij}) \quad \forall i \neq 1, \forall j \neq 1. \quad \text{——— (2)}$$

$$\Rightarrow u_i - u_j + n x_{ij} \le n-1, \quad \forall i \neq 1, \forall j \neq 1.$$

Recall: Projection of $P = \{(\bar{x},\bar{y}) \mid A\bar{x} + B\bar{y} \le \bar{b}\}$ to $\bar{x}$:

$$\mathrm{Proj}_{\bar{x}}(P) = \{\bar{x} \mid \exists \bar{y} : (\bar{x},\bar{y}) \in P\}.$$

$$u_i - u_j + n x_{ij} \le n-1 \quad \forall i \ne 1, \forall j \ne 1,$$

Can be written in ~~matrix~~ form as

$$nI\bar{x} + B\bar{u} \le (n-1)\,\bar{1}, \quad \text{where}$$

I is the identity matrix, B is the arc-node incidence matrix of G, with column for node 1 removed, and any non-zero entry corresponding to arcs incident to node 1 zeroed out, and $\bar{1}$ is the vector of ones.