MATH 567 : Lecture 9 (02/06/2025)

Today : * TSP : MTZ v/s subtour formulations
* sharp formulation of a disjunction

Recall : $\mathrm{Proj}_{\bar{x}}(P_{MTZ})$ to compare with $P_{subtour}$

$$u_i - u_j + n x_{ij} \le n-1 \quad \forall i \ne 1, \forall j \ne 1; \longrightarrow \text{matrix form is}$$

$$n I \bar{x} + B \bar{u} \le (n-1)\bar{1} \quad , \text{ where}$$

I is the identity matrix, B is the arc-node incidence matrix of G, with column for node 1 removed, and any non-zero entry corresponding to arcs incident to node 1 zeroed out, and $\bar{1}$ is the vector of ones.

$$\text{arcs}\left\{\begin{bmatrix} nI_m & \overbrace{B}^{\text{nodes, except 1}} \end{bmatrix}\right. \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \le \begin{bmatrix} n-1 \\ n-1 \\ \vdots \\ n-1 \end{bmatrix}$$

$$\begin{array}{cc} i & j \\ +1 & -1 \end{array} \qquad \bar{u}: u_2, \dots, u_n$$

The **node-arc incidence matrix** of a directed graph $G=(V,E)$ with $|V|=n$, $|E|=m$ is an $n \times m$ matrix with a row for each node and a column for each arc, with entries in $\{-1,0,1\}$. The column corresponding to arc (i,j) has a $+1$ in row i and a -1 in row j, and other entries are zero. B above is the transpose of this matrix, with the modifications made as specified.

Also, recall the definition of projection – we went from $A\bar{x} + B\bar{y} \le b$ to the space of $\bar{x}$ variables by eliminating the "unwanted" $\bar{y}$ variables.

Let $C = \{ \bar{v} \geq \bar{0} \mid \bar{v}^T B = \bar{0}^T \}$ be the projection cone.

$\bar{v}$?

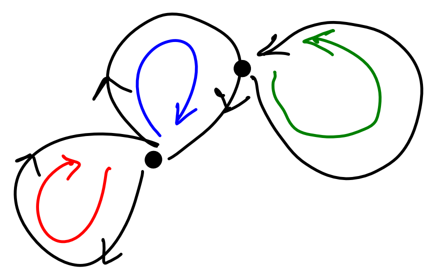
$$C = \{ \bar{v} \geq \bar{0} \mid \forall i \neq 1, \underbrace{\sum_j{}' v_{ij} = \sum_j{}' v_{ji}}_{\text{come in and go out at } i \text{ equal \# times}} \}$$

$= \{ \bar{v} \mid \bar{v}$ is a circulation in $G \}$

or circuit, generalization of a cycle

$\bar{v}$ is the incidence vector of a circulation.

$\bar{v} \in \{0,1\}^m$, $m = \#$ arcs.



It turns out we can describe all circulations as unions of a finite set of "basic" cycles. In other words, the projection cone is finitely generated.

$$C = \{ \sum_{i=1}^{k} \lambda_i \bar{v}^i \mid \lambda_i \geq 0 \} \quad \text{where}$$

$\bar{v}^1, \dots, \bar{v}^k$ are the incidence vectors of a set of basic cycles.

$$\Rightarrow \text{Proj}_{\bar{x}}(P_{MTZ}) = \left\{ \bar{x} \;\middle|\; \begin{array}{l} (\bar{v}^i)^T (n\bar{I})\bar{x} \le (\bar{v}^i)^T (n-1)\bar{1},\ i=1,\dots,k; \\ \text{and system (1)} \end{array} \right\}.$$

We do not get any inequalities stronger than $\triangle$ here.

If $(\bar{v}^i)^T = [0, 0, \dots, 0, \underbrace{1, 1 \dots, 1}, 0, \dots, 0]$,

1's for $(i,j) \in C$; assumed to be set all together here WLOG.

then

$$(\bar{v}^i)^T (n\bar{I})\bar{x} \le (\bar{v}^i)^T (n-1)\bar{1}$$

$$n\, x(C) \le (n-1)\, |C|$$

$$\Rightarrow \quad x(C) \le \left(1 - \frac{1}{n}\right)|C|, \text{ which is } \triangle.$$

$\Rightarrow$ The subtour formulation is stronger!

Sharp Formulation of a Disjunction

Let $S = Q_1 \cup \cdots \cup Q_k$ where $Q_i = \{\bar{x} \mid A_i \bar{x} \le \bar{b}^i\}$, $i=1,\dots,k$.
$\bar{x} \in \mathbb{R}^n$, are non-empty polyhedra with the same recession cone. Then $P \subseteq \underbrace{\mathbb{R}^n}_{\bar{x}} \times \underbrace{\mathbb{R}^k}_{\bar{y}} \times \underbrace{(\mathbb{R}^n \times \cdots \times \mathbb{R}^n)}_{k \text{ copies, for } \bar{x}^1,\dots,\bar{x}^k}$

defined as the set of all vectors $(\bar{x}, \bar{y}, \bar{x}^1, \dots, \bar{x}^k)$ that satisfy

$$P = \left\{ \begin{array}{c} A_1 \bar{x}^1 \le \bar{b}^1 y_1 \\ \vdots \qquad \vdots \\ A_k \bar{x}^k \le \bar{b}^k y_k \\ \bar{x}^1 + \cdots + \bar{x}^k = \bar{x} \\ y_1 + \cdots + y_k = 1 \\ 0 \le y_i \le 1 \;\; \forall i \end{array} \right\}$$

could ignore, as $y_1 + \cdots + y_k = 1$ implies the same with $y_i \ge 0$.

is a sharp formulation for S.

S is the same (*) set for which we wrote (*-big-M) and (*-sharp) formulations.

Q: How do we prove P is indeed a sharp formulation?

To show P is a sharp formulation for S, we have to show $\text{Proj}_{\bar{x}}(P) = \text{conv}(S)$.

We need $\mathrm{conv}\left(\bigcup_{i=1}^{k} Q_i\right)$ to be closed, but we'll assume that.

It appears the approach to identify all corner points will not work here. Could we use another approach?

Def An inequality $\bar{a}^T\bar{x} \le \beta$ is a **valid inequality** for $X \subseteq \mathbb{R}^n$ if $\bar{a}^T\bar{x} \le \beta \;\; \forall \bar{x} \in X$.

We can try to derive conditions that guarantee an inequality is valid for $\mathrm{conv}(S)$ iff it is valid for $\mathrm{Proj}_{\bar{x}}(P)$.

$$\bar{a}^T\bar{x} \le \beta \text{ is valid for } \mathrm{conv}(Q_1 \cup \dots \cup Q_k)$$

$$\iff \bar{a}^T\bar{x} \le \beta \text{ is valid for each of } Q_1, \dots, Q_k.$$

$$\iff \exists\, \bar{u}^i \ge \bar{0}, \quad \bar{a}^T = (\bar{u}^i)^T A_i, \quad (\bar{u}^i)^T \bar{b}^i \le \beta, \text{ i.e.,}$$

we can derive $\bar{a}^T\bar{x} \le \beta$ from $A_i\bar{x} \le \bar{b}^i$.

$\bar{a}^T\bar{x} \leq \beta$ is valid for $\mathrm{Proj}_{\bar{x}}(P) \iff \exists$ multipliers that derive $\bar{a}^T\bar{x} \leq \beta$ from P by eliminating $\bar{x}^1, \dots, \bar{x}^k, y_1, \dots, y_k$.

$$
\begin{array}{l|cccccccccc}
\bar{a}^T & \bar{x} & -\bar{x}^1 & -\bar{x}^2 & \cdots & -\bar{x}^k & & & & & = \bar{0} \\
\bar{v}^{1T} & & A_1\bar{x}^1 & & & & -\bar{b}^1 y_1 & & & & \leq \bar{0} \\
\bar{v}^{2T} & & & A_2\bar{x}^2 & & & & -\bar{b}^2 y_2 & & & \leq \bar{0} \\
\vdots & & & & \ddots & & & & & & \vdots \\
\bar{v}^{kT} & & & & & A_k\bar{x}^k & & & & -\bar{b}^k y_k & \leq \bar{0} \\
\beta' & & & & & & y_1 & +\, y_2 & + \cdots + & y_k & = 1 \\
\beta'_1 & & & & & & -y_1 & & & & \leq 0 \\
\beta'_2 & & & & & & & -y_2 & & & \leq 0 \\
\vdots & & & & & & & & \ddots & & \vdots \\
\beta'_k & & & & & & & & & -y_k & \leq 0
\end{array}
$$

We need

$$\bar{a}^T = (\bar{v}^1)^T A_1 \to \text{eliminates } \bar{x}^1$$

$$\vdots$$

$$\bar{a}^T = (\bar{v}^k)^T A_k \to \text{eliminates } \bar{x}^k$$

We need $\bar{a} \geq \bar{0}$, $\bar{v}^i \geq \bar{0}$, $\beta' \geq 0$, $\beta'_i \geq 0$ and $\beta' \leq \beta$

and

$$(-\bar{v}^1)^T\bar{b}^1 + \beta' - \beta'_1 = 0 \to \text{eliminate } y_1$$

$$\vdots$$

$$(-\bar{v}^k)^T\bar{b}^k + \beta' - \beta'_k = 0 \to \text{eliminate } y_k$$

$$\left\} \begin{array}{c} (-\bar{v}^1)^T\bar{b}^1 + \beta' \geq 0 \\ \vdots \\ (-\bar{v}^k)^T\bar{b}^k + \beta' \geq 0 \end{array} \right.$$

Notice we need $\beta_i' \geq 0$, and hence could scale the right-hand sides of these inequalities to get rid of β_i's.

$$\left\{ \begin{array}{ll} \bar{a}^T = (\bar{v}^i)^T A_i, & i = 1, \dots, k \\ \beta' \geq (\bar{v}^i)^T \bar{b}^i, & i = 1, \dots, k \\ \beta' \leq \beta & \end{array} \right\}$$

Need to show this system has non-negative solution in $(\bar{v}^i, \beta')$.

We could use this approach for specific instances in which the A_i and $\bar{b}^i$ are provided.

Definitions and Results on Polyhedra

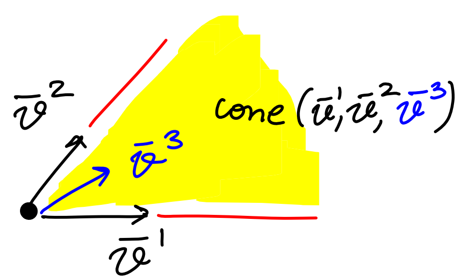
We collect several relevant definitions and results related to polyhedra here. We will use these results in further elucidating properties and strengths of formulations, as well as comparing them.

* $C \subseteq \mathbb{R}^n$ is convex if $\lambda\bar{x} + (1-\lambda)\bar{y} \in C \;\; \forall \bar{x}, \bar{y} \in C$, $\lambda \in [0,1]$.

* $C \subseteq \mathbb{R}^n$ is a convex cone if $\lambda\bar{x} + \mu\bar{y} \in C \;\; \forall \bar{x}, \bar{y} \in C$, and $\lambda, \mu \geqslant 0$.

* $\text{cone}(\{\bar{v}^1, \dots, \bar{v}^k\}) = \{\bar{x} \mid \bar{x} = \sum_{i=1}^{k} \lambda_i \bar{v}^i,\ \lambda_i \geqslant 0 \;\forall i\}$.

  the smallest cone containing $\bar{v}^1, \dots, \bar{v}^k$.
  k is finite $\Longrightarrow$ C is a finitely generated cone.



* A cone C is polyhedral if $C = \{\bar{x} \mid A\bar{x} \leq \bar{0}\}$. Here, C is the intersection of finitely many linear half-spaces. $\{\bar{x} \mid \bar{a}^T\bar{x} \leq 0\}$

* A convex cone is polyhedral iff it is finitely generated.