Flat Norm Decomposition of Integral Currents

Bala Krishnamoorthy

with

Sharif Ibrahim, Kevin Vixie

Washington State University

preprint on arXiv

CURRENTS

* generalized surfaces with multiplicities & orientation
  — in GMT; isoperimetric problems, soap bubble conjectures

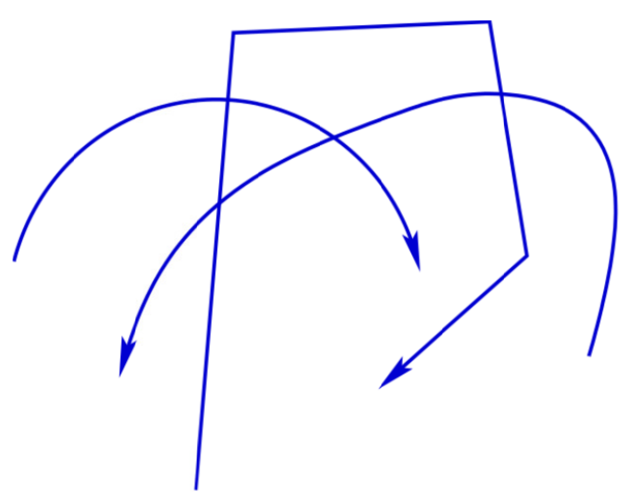
CURRENTS

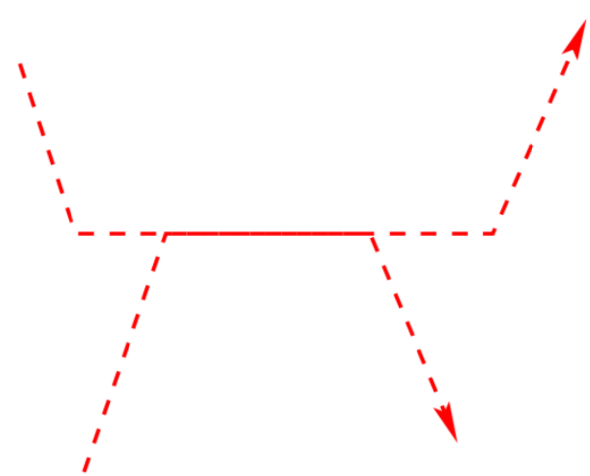
* generalized surfaces with multiplicities & orientation
  - in GMT; isoperimetric problems, soap bubble conjectures
  - we consider integer multiplicities

CURRENTS

* generalized surfaces with multiplicities & orientation
  - in GMT; isoperimetric problems, soap bubble conjectures
  - we consider integer multiplicities

e.g., 1D-current: collection of oriented curves in $\mathbb{R}^d$

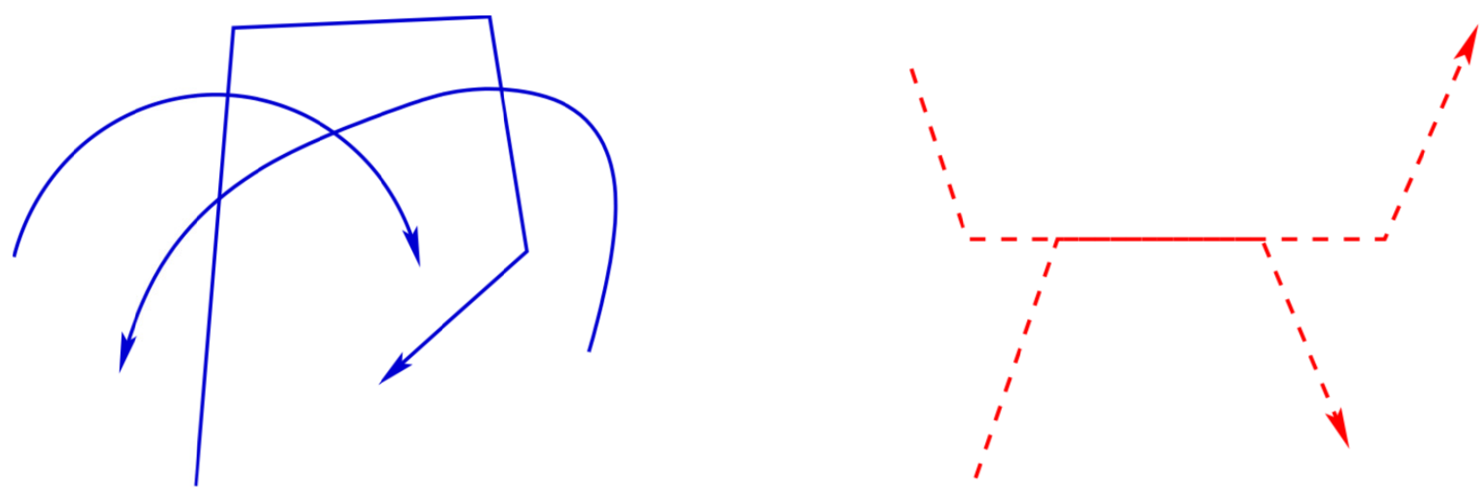




CURRENTS

* generalized surfaces with multiplicities & orientation
  - in GMT; isoperimetric problems, soap bubble conjectures
  - we consider integer multiplicities

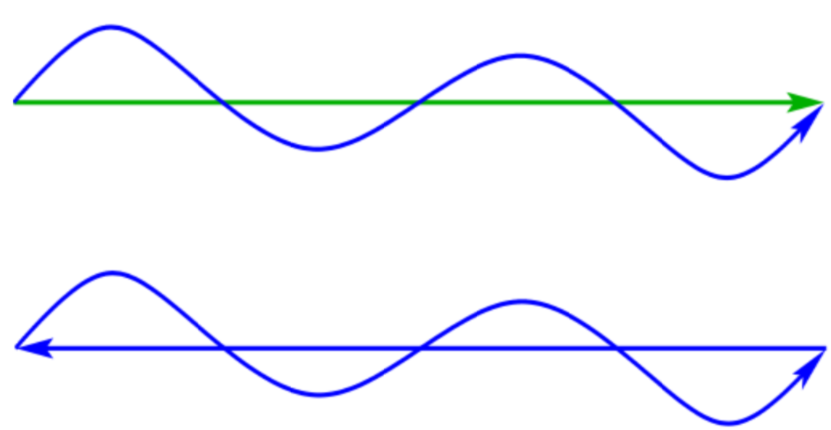
  e.g., 1D-current: collection of oriented curves in $\mathbb{R}^d$



* mass of d-current = its weighted d-volume
  - we assume finite mass

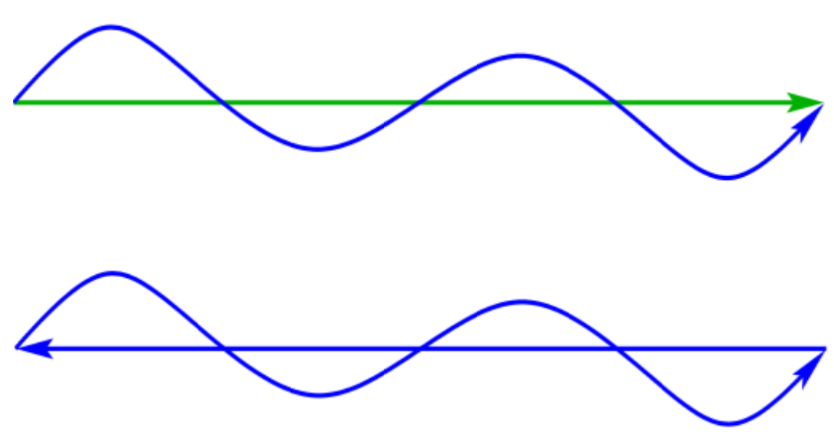
DISTANCE BETWEEN CURRENTS

* Hausdorff, Fréchet problematic



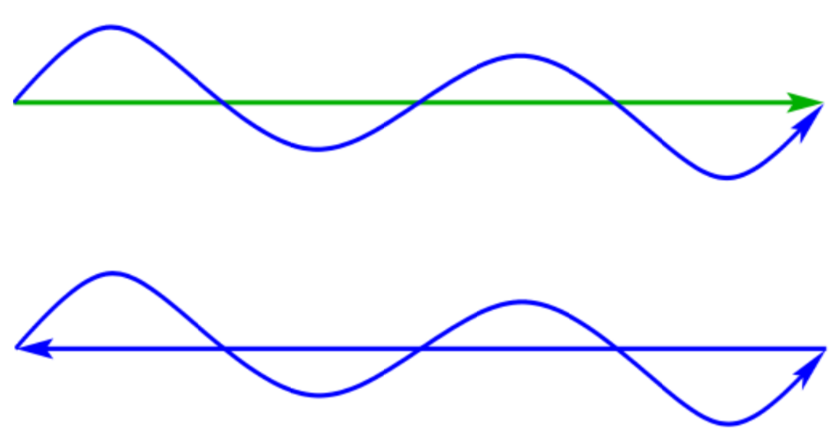
Distance Between Currents

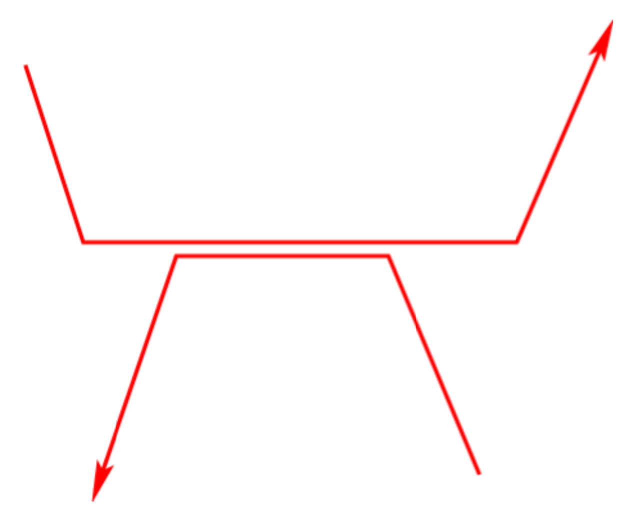
* Hausdorff, Fréchet problematic
* mass of difference $M(T_1 - T_2)$?



Distance Between Currents

* Hausdorff, Fréchet problematic
* mass of difference $M(T_1 - T_2)$?
  - does not always fit intuition
  - sensitivite to small changes in T_1, T_2



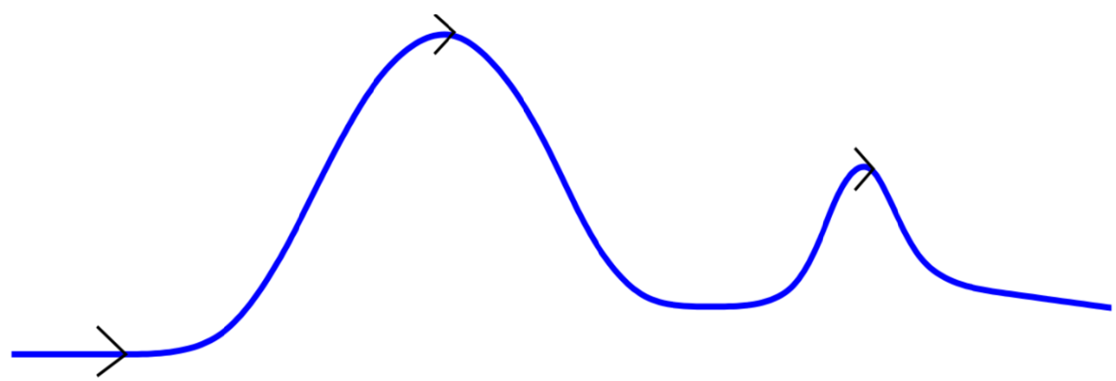


FLAT NORM

* $\mathbb{F}(T) = \min_S \{ \mathbb{M}_d(T - \partial S) + \mathbb{M}_{d+1}(S) \mid S \text{ is } (d+1)\text{-current} \}$

FLAT NORM

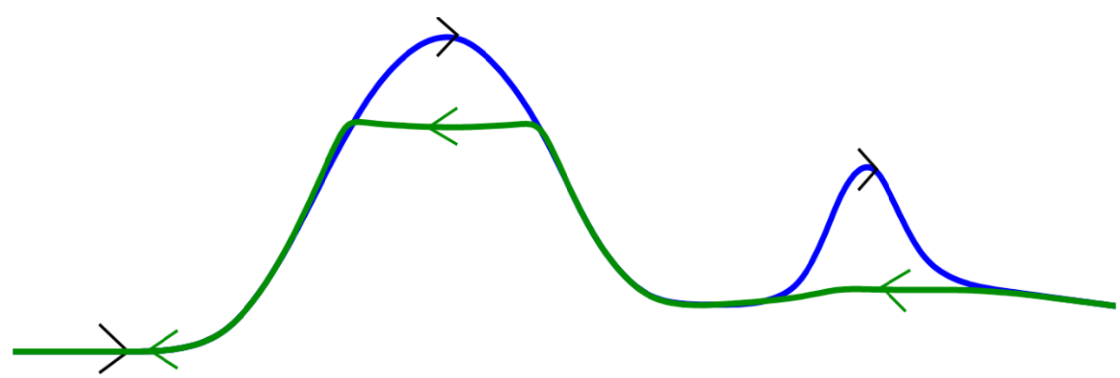
- $\mathbb{F}(T) = \min_S \{ M_d(T - \partial S) + M_{d+1}(S) \mid S \text{ is } (d+1)\text{-current} \}$
- Idea (1D): How to erase T at min cost?



FLAT NORM

* $\mathbb{F}(T) = \min_S \{ M_d(T - \partial S) + M_{d+1}(S) \mid S \text{ is } (d+1)\text{-current} \}$
* Idea (1D): How to erase T at min cost?

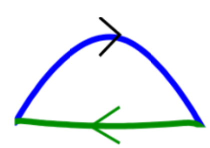
operations: Add 1-curve; cost = length

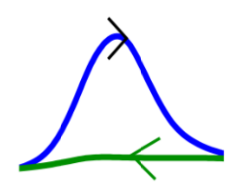


FLAT NORM

* $\mathbb{F}(T) = \min_S \{ M_d(T - \partial S) + M_{d+1}(S) \mid S \text{ is } (d+1)\text{-current} \}$
* Idea (1D): How to erase T at min cost?

Operations: Add 1-curve; cost = length

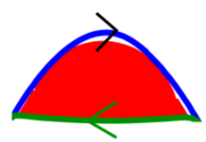
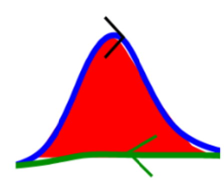




FLAT NORM

* $\mathbb{F}(T) = \min\limits_{S} \{ M_d(T - \partial S) + M_{d+1}(S) \mid S \text{ is } (d+1)\text{-current} \}$
* Idea (1D): How to erase T at min cost?

Operations: Add 1-curve; cost = length
Trace boundary of 2D region; cost = area

FLAT NORM

* $\mathbb{F}(T) = \min_S \{ M_d(T - \partial S) + M_{d+1}(S) \mid S \text{ is } (d+1)\text{-current} \}$
* Idea (1D): How to erase T at min cost?

Operations: Add 1-curve; cost = length
Trace boundary of 2D region; cost = area

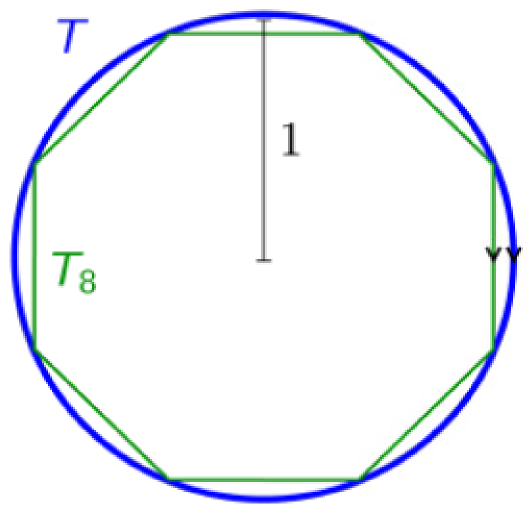
$\mathbb{F}(T)$ = minimum such total cost

EXAMPLE

* flat distance between P, Q: $\mathbb{F}(P,Q) = \mathbb{F}(P-Q)$

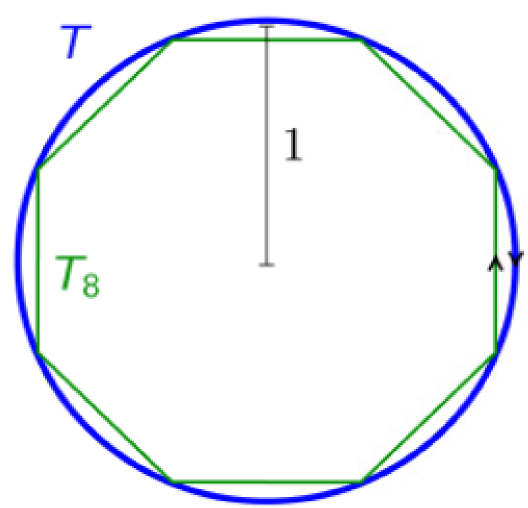
EXAMPLE

* flat distance between P, Q: $\mathbb{F}(P,Q) = \mathbb{F}(P-Q)$
* T: unit circle (in $\mathbb{R}^2$), T_n: inscribed n-gon
  both oriented clockwise



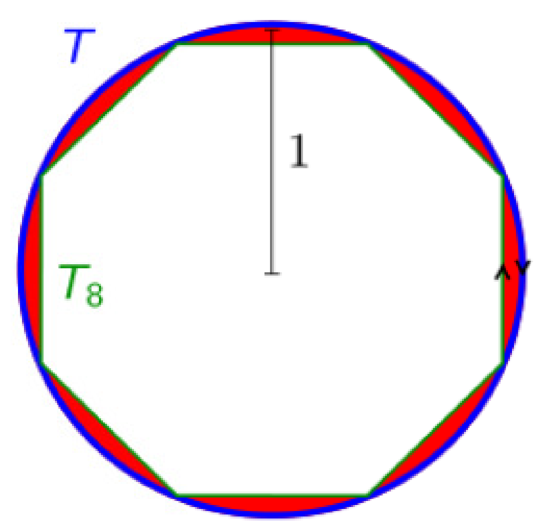
EXAMPLE

- flat distance between P, Q: $\mathbb{F}(P,Q) = \mathbb{F}(P-Q)$
- T: unit circle (in $\mathbb{R}^2$), T_n: inscribed n-gon
  both oriented clockwise



EXAMPLE

- flat distance between P, Q: $\mathbb{F}(P,Q) = \mathbb{F}(P-Q)$
- T: unit circle (in $\mathbb{R}^2$), T_n: inscribed n-gon
  both oriented clockwise



- as $n \to \infty$, $\mathbb{M}(T-T_n) \to 4\pi$, but $\mathbb{F}(T, T_n) \to 0$.

INTEGRAL CURRENTS

* oriented rectifiable sets, integer multiplicities, finite mass, boundary has same properties

INTEGRAL CURRENTS

* oriented rectifiable sets, integer multiplicities, finite mass, boundary has same properties

Q: When is the flat norm decomposition of an integral current also integral?

INTEGRAL CURRENTS

* oriented rectifiable sets, integer multiplicities, finite mass, boundary has same properties

Q: When is the flat norm decomposition of an integral current also integral?

- YES for codimension-1 boundaries (L^1TV functional - Morgan & Vixie, 2007)

INTEGRAL CURRENTS

* oriented rectifiable sets, integer multiplicities, finite mass, boundary has same properties

Q: When is the flat norm decomposition of an integral current also integral?

- YES for codimension-1 boundaries (L^1TV functional - Morgan & Vixie, 2007)

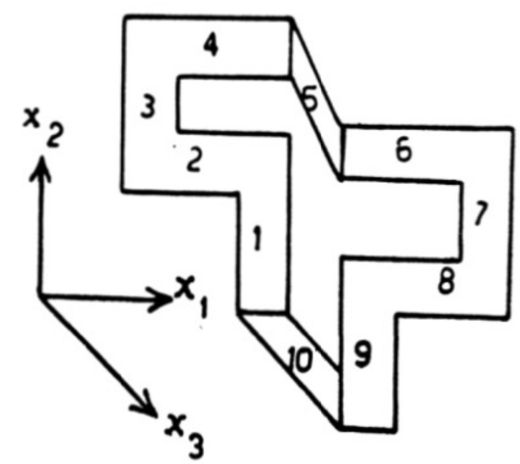
✓ analysis framework: simplicial to continuous

- YES for d-currents in $\mathbb{R}^{d+1}$ if a triangulation result holds
- YES in 2D

RELATED WORK

* Area-minimizing fillings - cheaper by the dozen!
  - L. Young (1963), F. Morgan (1984), B. White (1984)
  - closed curves C in $\mathbb{R}^4$

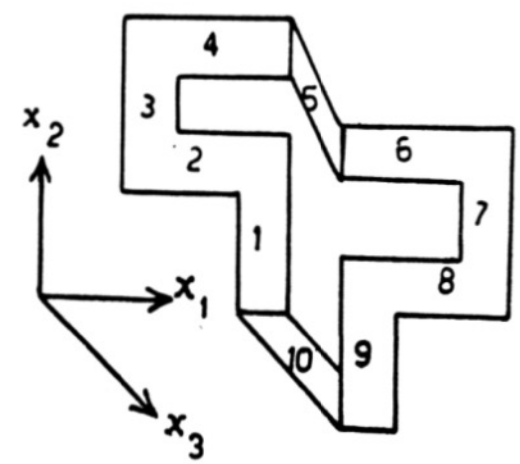
Area filling 2C < 2(Area filling C)



RELATED WORK

* Area-minimizing fillings - cheaper by the dozen!
  - L. Young (1963), F. Morgan (1984), B. White (1984)
  - closed curves C in $\mathbb{R}^4$

    Area filling $2C$ <
    2(Area filling C)

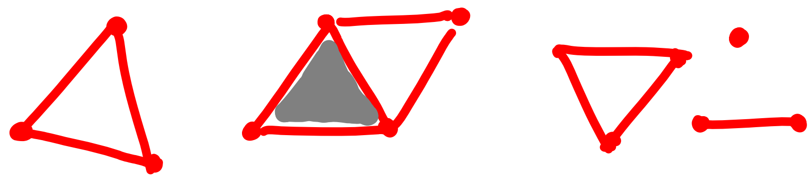


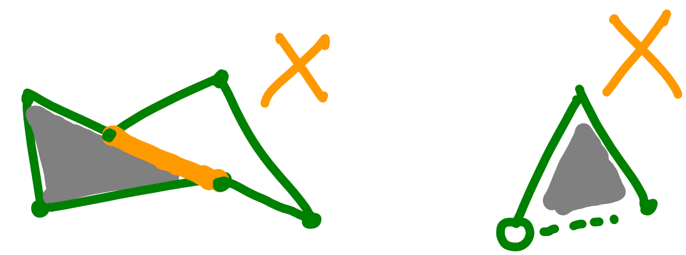
* Open problem considered by F. Almgren (White 1998): If $2T_i$ is a sequence of integral flat chains that converge in integral flat topology, must T_i also converge?

SIMPLICIAL FLAT NORM

— Ibrahim, K, Vixie (2013)

* discretize the problem on a <u>simplicial complex</u>
  - a collection of simplices that includes all faces, and intersections are faces



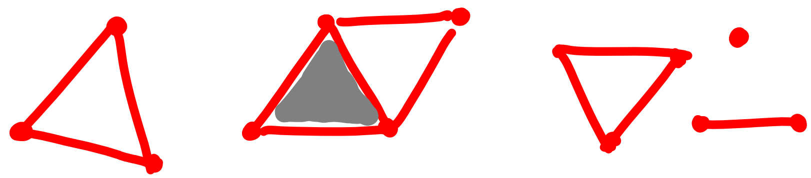


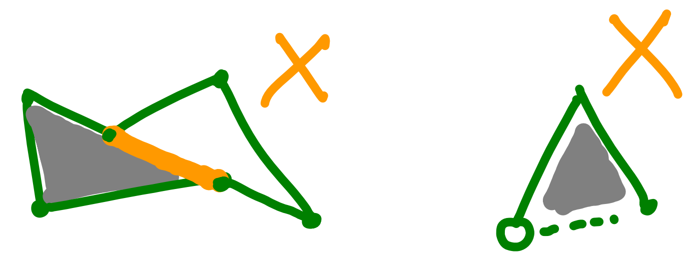
not simplicial complexes

SIMPLICIAL FLAT NORM

— Ibrahim, K, Vixie (2013)

* discretize the problem on a <u>simplicial complex</u>
  - a collection of simplices that includes all faces, and intersections are faces





not simplicial complexes

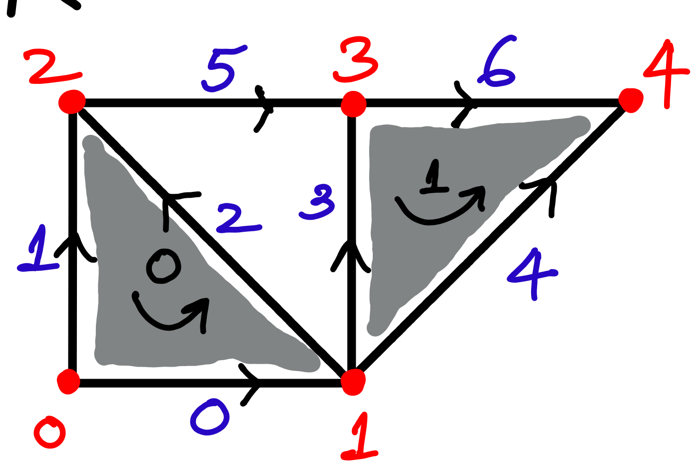
* currents are chains on the simplicial complex

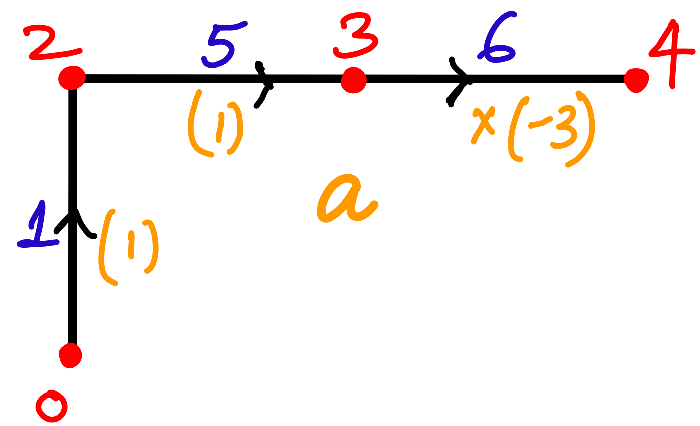
CHAINS

chains on a simplicial complex

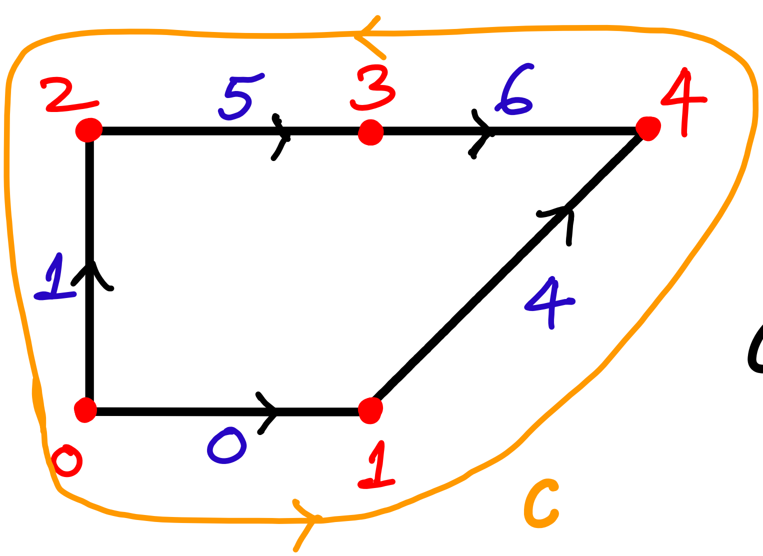
1-chains

K

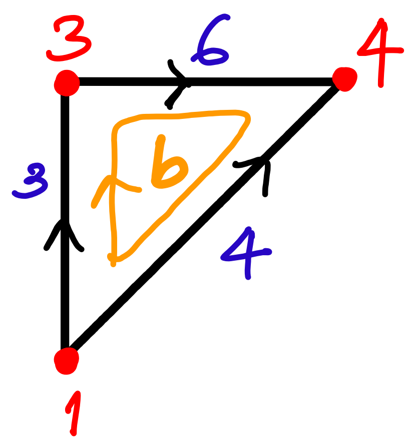




$$a = \begin{array}{c} 0\\1\\2\\3\\4\\5\\6 \end{array}\begin{bmatrix} 0\\1\\0\\0\\0\\1\\-3 \end{bmatrix}$$



$$c = \begin{array}{c} 0\\1\\2\\3\\4\\5\\6 \end{array}\begin{bmatrix} 1\\-1\\0\\0\\1\\-1\\-1 \end{bmatrix}$$

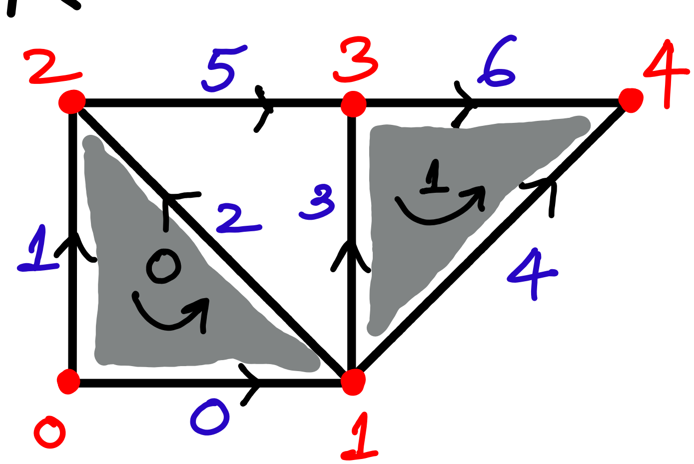


$$b = \begin{array}{c} 0\\1\\2\\3\\4\\5\\6 \end{array}\begin{bmatrix} 0\\0\\0\\1\\-1\\0\\1 \end{bmatrix}$$

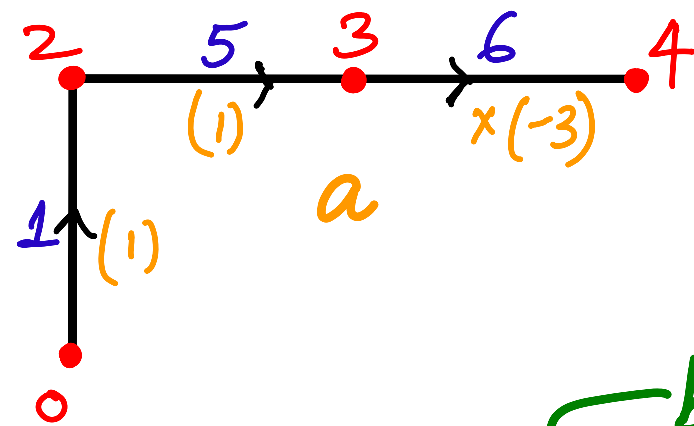
CHAINS

chains on a simplicial complex

K

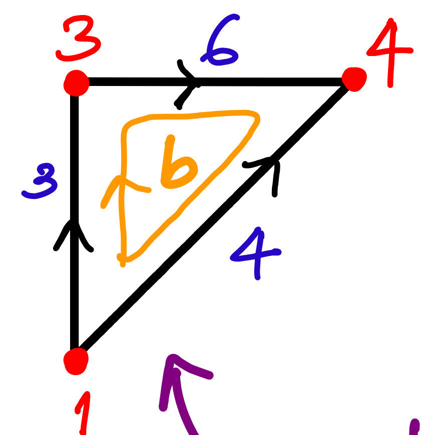


1-chains

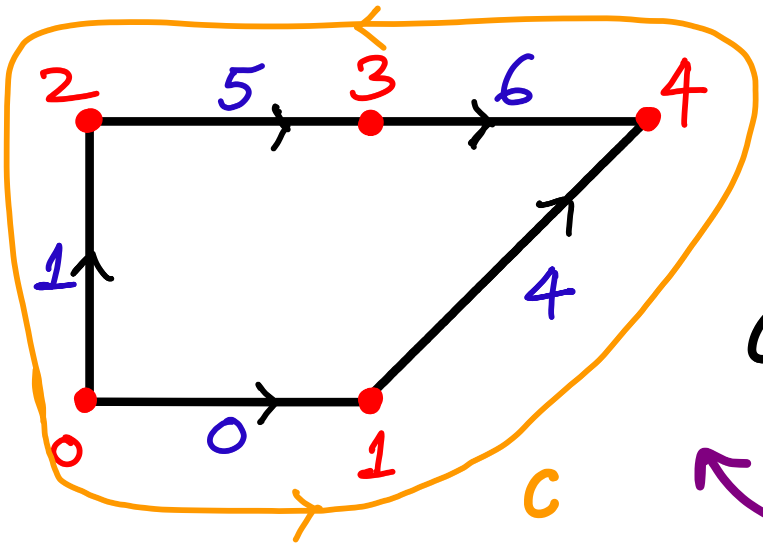


$$a = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

boundary



$$b = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$



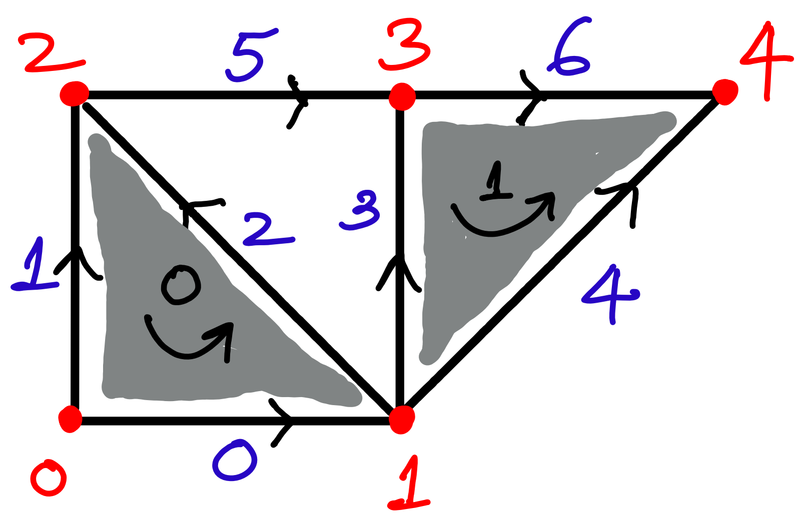
$$c = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

cycles

Boundary Matrix $[\partial_{p+1}]$

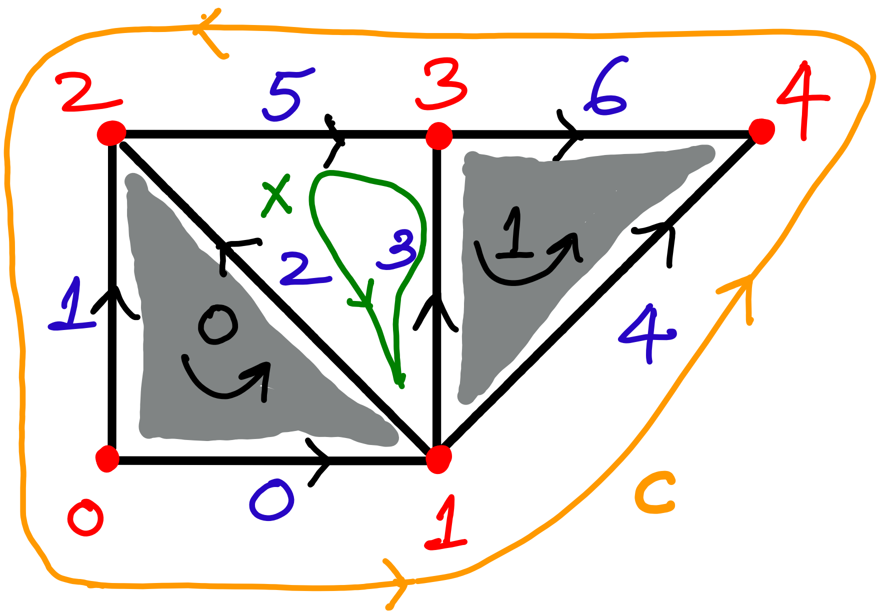
$$\partial_{p+1} : C_{p+1}(K) \to C_p(K)$$

With m p-simplices and n $(p+1)$-simplices in K, $[\partial_{p+1}]$ is an $m \times n$ matrix with entries in $\{-1, 0, 1\}$.



$$[\partial_2] = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \end{array}$$

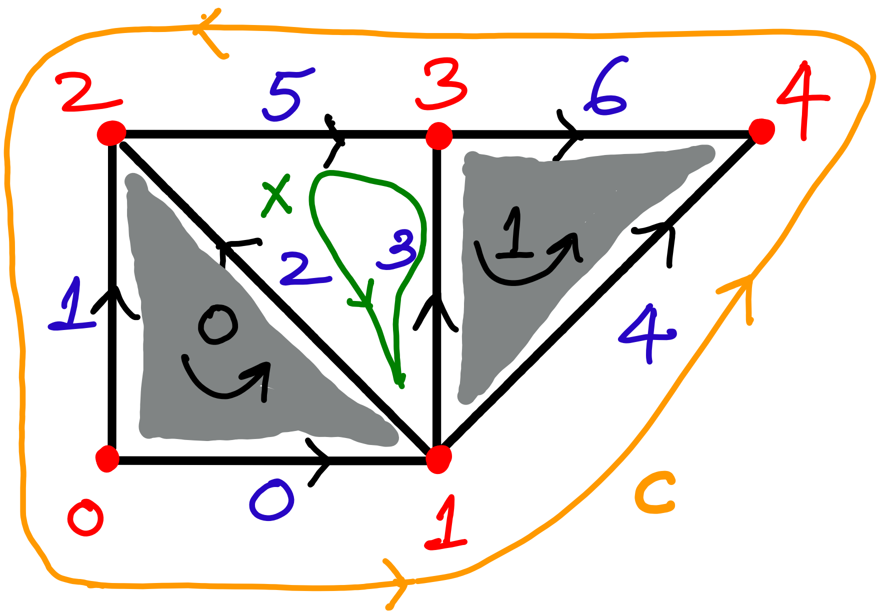
HOMOLOGOUS CHAINS



x is homologous to c

→ both go around the same hole

HOMOLOGOUS CHAINS



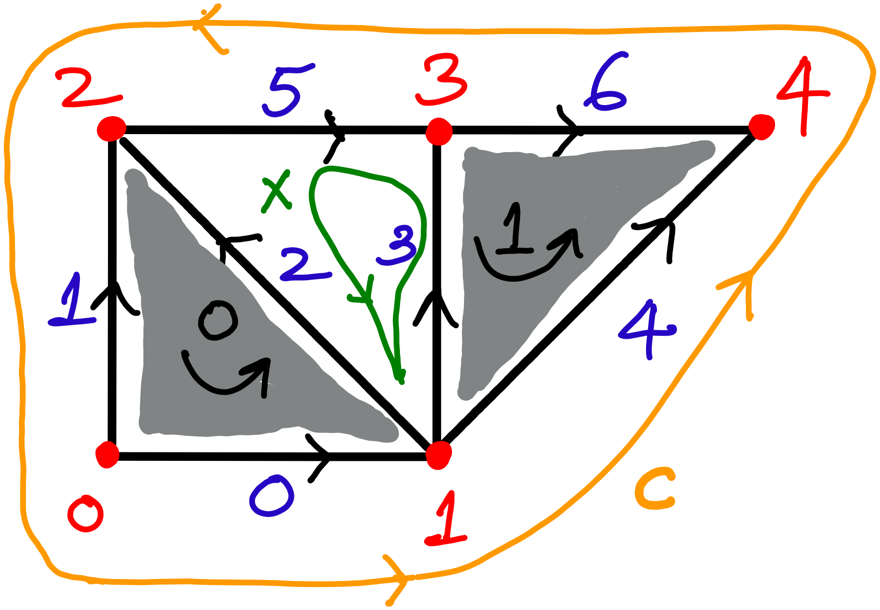
x is homologous to c

→ both go around the same hole

$$c = \begin{array}{c} 0\\1\\2\\3\\4\\5\\6 \end{array}\begin{bmatrix} 1\\-1\\0\\0\\1\\-1\\-1 \end{bmatrix}, \quad x = \begin{array}{c} 0\\1\\2\\3\\4\\5\\6 \end{array}\begin{bmatrix} 0\\0\\-1\\1\\0\\-1\\0 \end{bmatrix}, \quad [\partial_2] = \begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1 & 0\\-1 & 0\\1 & 0\\0 & -1\\0 & 1\\0 & 0\\0 & -1 \end{bmatrix} \end{array}$$

$$x = c - [\partial_2]\begin{bmatrix} 1\\1 \end{bmatrix}$$

HOMOLOGOUS CHAINS



x is homologous to c

$\rightarrow$ both go around the same hole

$$c = \begin{array}{c} 0\\1\\2\\3\\4\\5\\6 \end{array}\begin{bmatrix} 1\\-1\\0\\0\\1\\-1\\-1 \end{bmatrix}, \quad x = \begin{array}{c} 0\\1\\2\\3\\4\\5\\6 \end{array}\begin{bmatrix} 0\\0\\-1\\1\\0\\-1\\0 \end{bmatrix}, \quad [\partial_2] = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 0\\ 1 & -1 & 0\\ 2 & 1 & 0\\ 3 & 0 & -1\\ 4 & 0 & 1\\ 5 & 0 & 0\\ 6 & 0 & -1 \end{array}$$

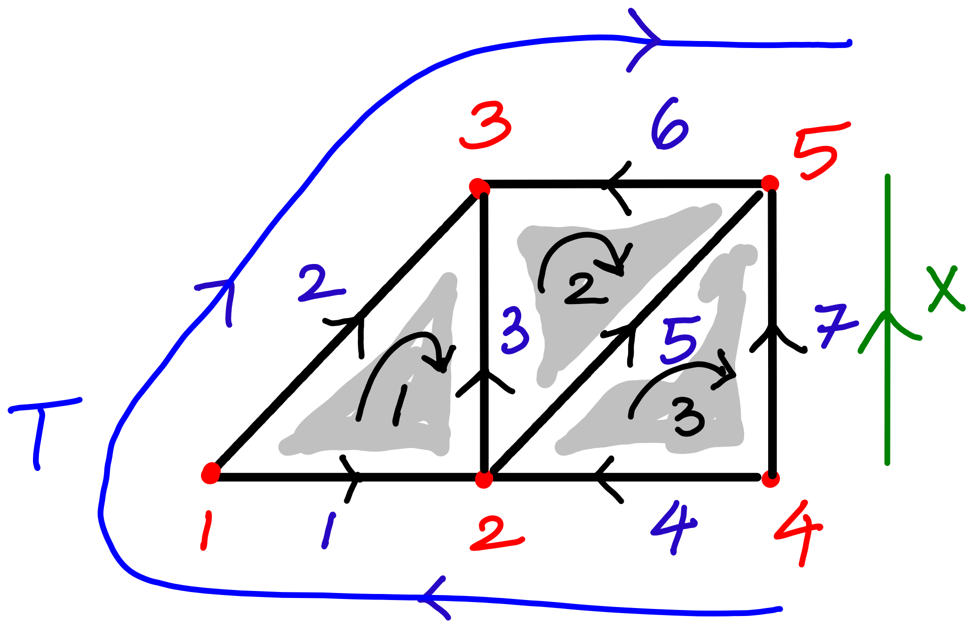
$$x = c - [\partial_2]\begin{bmatrix} 1\\1 \end{bmatrix}$$

In general,

$$\mathbf{x} = \mathbf{c} - [\partial_{p+1}]\mathbf{y}$$

for $y \in \mathbb{Z}^n$

Flat Norm Decomposition



$$t = \begin{array}{c}1\\2\\3\\4\\5\\6\\7\end{array}\begin{bmatrix}-1\\1\\0\\1\\0\\-1\\0\end{bmatrix} \qquad x = \begin{bmatrix}0\\0\\0\\0\\0\\0\\1\end{bmatrix}$$

x is homologous to t: $x = t - [\partial_2]s$ for

$$[\partial_2] = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ 4 & 0 & 0 & 1 \\ 5 & 0 & -1 & 1 \\ 6 & 0 & -1 & 0 \\ 7 & 0 & 0 & -1 \end{array} \qquad s = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

SFN as an Integer Program

$$\min \sum_{i=1}^{m} w_i |x_i| + \sum_{j=1}^{n} v_j |s_j|$$

$$x = t - [\partial_{p+1}] s, \quad x \in \mathbb{Z}^m, \; s \in \mathbb{Z}^n$$

SFN AS AN INTEGER PROGRAM

piecewise linear; $w_i, v_j \geq 0$

$$\min \sum_{i=1}^{m} w_i |x_i| + \sum_{j=1}^{n} v_j |s_j|$$

$$x = t - [\partial_{p+1}] s, \quad x \in \mathbb{Z}^m, \; s \in \mathbb{Z}^n$$

$$\min \sum_{i=1}^{m} w_i (x_i^+ + x_i^-) + \sum_{j=1}^{n} v_j (s_j^+ + s_j^-) \qquad \text{(IP)}$$

s.t. $\quad x^+ - x^- = t - [\partial_{p+1}](s^+ - s^-)$

$\quad x^+, x^- \geq 0$

$\quad s^+, s^- \geq 0 \qquad x^+, x^- \in \mathbb{Z}^m, \; s^+, s^- \in \mathbb{Z}^n$

SFN as an Integer Program

$$\min \sum_{i=1}^{m} w_i |x_i| + \sum_{j=1}^{n} v_j |s_j|$$

piecewise linear; $w_i, v_j \geq 0$

$$x = t - [\partial_{p+1}]s, \quad x \in \mathbb{Z}^m, \; s \in \mathbb{Z}^n$$

$$\min \sum_{i=1}^{m} w_i (x_i^+ + x_i^-) + \sum_{j=1}^{n} v_j (s_j^+ + s_j^-) \qquad \text{(IP)}$$

s.t. $x^+ - x^- = t - [\partial_{p+1}](s^+ - s^-)$

$x^+, x^- \geq 0$

$s^+, s^- \geq 0$

$\boxed{x^+, x^- \in \mathbb{Z}^m, \; s^+, s^- \in \mathbb{Z}^n}$

ignore to get LP relaxation

SFN AND TU OF $[\partial_{p+1}]$

$$\begin{aligned} \min \ & \sum_{i=1}^{m} w_i (x_i^+ + x_i^-) + \sum_{j=1}^{n} v_j (s_j^+ + s_j^-) \\ \text{s.t.} \ & x^+ - x^- = t - [\partial_{p+1}](s^+ - s^-) \\ & x^+, x^- \geqslant 0, \ s^+, s^- \geqslant 0 \end{aligned} \qquad \text{(LP)}$$

* The constraint matrix of above LP is TU iff $[\partial_{p+1}]$ is TU.

SFN AND TU OF $[\partial_{p+1}]$

$$\begin{aligned} \min \ & \sum_{i=1}^{m} w_i (x_i^+ + x_i^-) + \sum_{j=1}^{n} v_j (s_j^+ + s_j^-) \\ \text{s.t. } & x^+ - x^- = t - [\partial_{p+1}](s^+ - s^-) \qquad \text{(LP)} \\ & x^+, x^- \geq 0, \ s^+, s^- \geq 0 \end{aligned}$$

- The constraint matrix of above LP is TU iff $[\partial_{p+1}]$ is TU.
- $[\partial_{p+1}]$ is TU for K in $\mathbb{R}^{d+1}$ (Dey, Hirani, K, 2010)

SFN AND TU OF $[\partial_{p+1}]$

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} w_i (x_i^+ + x_i^-) + \sum_{j=1}^{n} v_j (s_j^+ + s_j^-) \\ \text{s.t.} \quad & x^+ - x^- = t - [\partial_{p+1}](s^+ - s^-) \\ & x^+, x^- \geq 0, \; s^+, s^- \geq 0 \end{aligned} \qquad \text{(LP)}$$

- The constraint matrix of above LP is TU iff $[\partial_{p+1}]$ is TU.
- $[\partial_{p+1}]$ is TU for K in $\mathbb{R}^{d+1}$ (Dey, Hirani, K, 2010)

✓ integral in = integral out for codimension-1 simplicial currents

INTEGRAL IN = INTEGRAL OUT?

* Can we use the simplicial result to obtain the continuous result?

INTEGRAL IN = INTEGRAL OUT ?

* Can we use the simplicial result to obtain the continuous result ?

? Could we take simplicial approximations of T and somehow take the limit of its simplicial flat norm decompositions to get a continuous decomposition ?

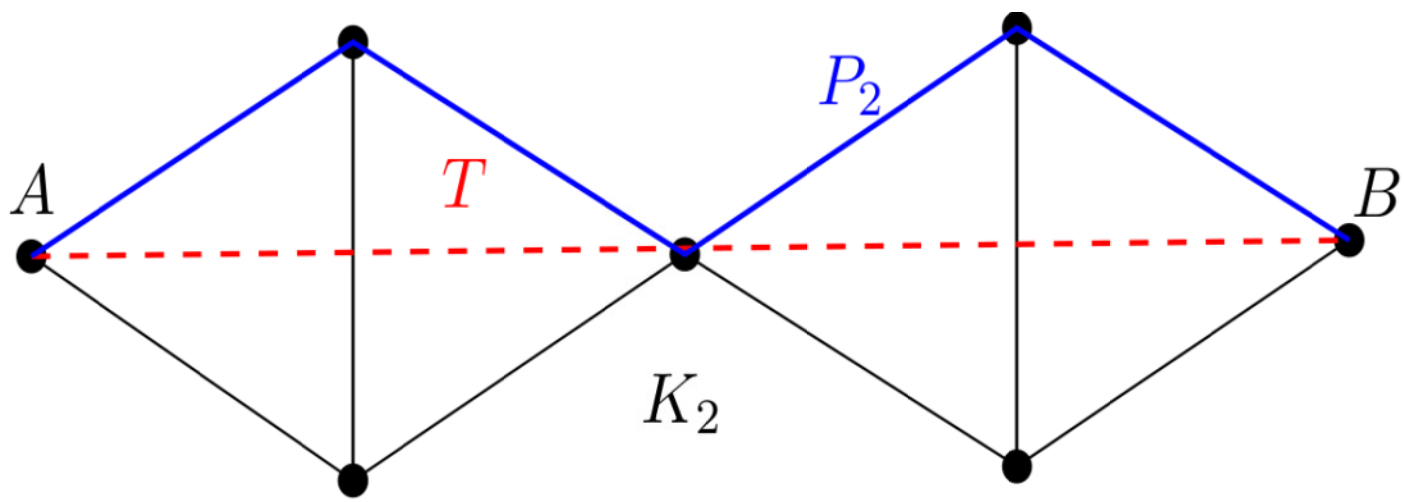
INTEGRAL IN = INTEGRAL OUT?

* Can we use the simplicial result to obtain the continuous result?

? Could we take simplicial approximations of T and somehow take the limit of its simplicial flat norm decompositions to get a continuous decomposition?

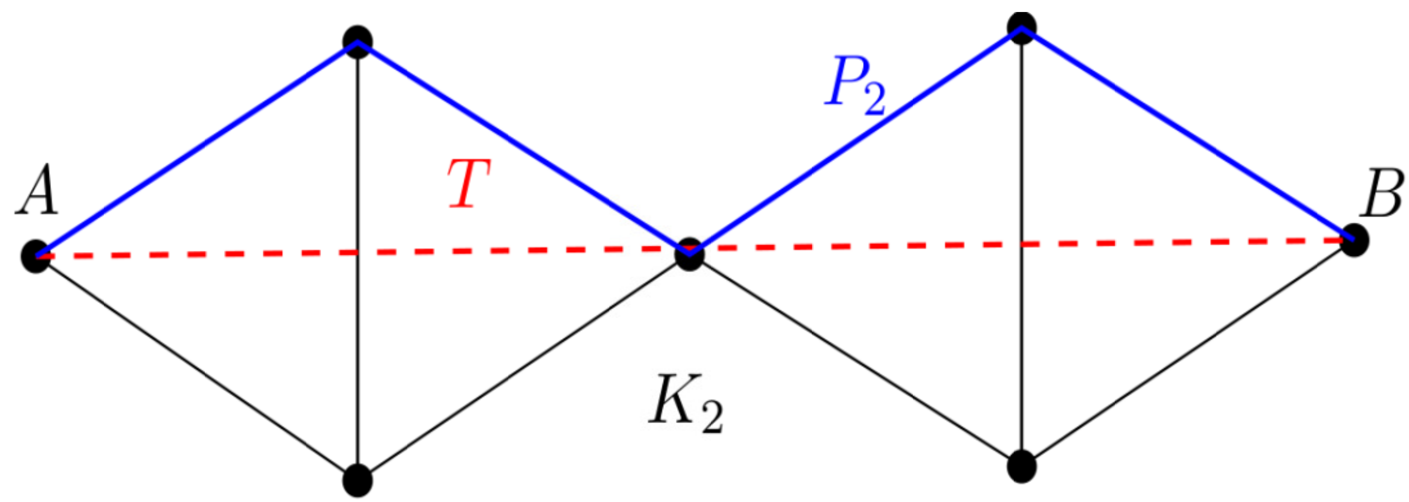
not straightforward...

SFN $\not\to$ FLAT NORM



K_n : $2n$ equilateral triangles

SFN $\not\to$ FLAT NORM

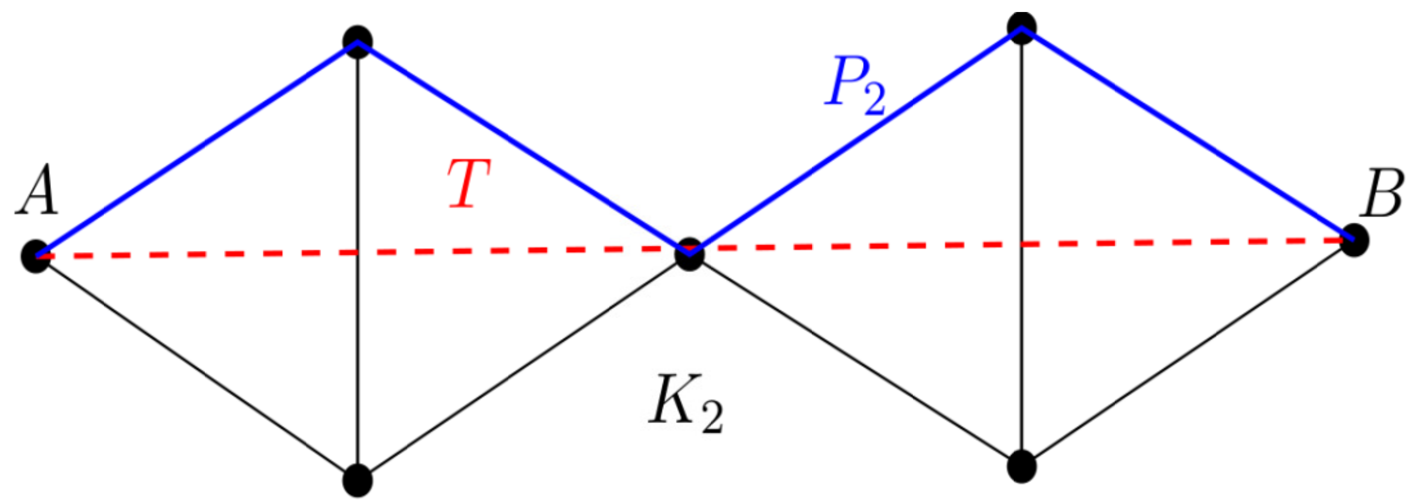


K_n : $2n$ equilateral triangles

$\mathbb{F}(T - P_n) \to 0$ as $n \to \infty$, but

$\mathbb{F}_{K_n}(P_n) = \frac{2}{\sqrt{3}} \mathbb{F}(T) \not\to \mathbb{F}(T)$

SFN $\not\to$ FLAT NORM



K_n : $2n$ equilateral triangles

$\mathbb{F}(T-P_n) \to 0$ as $n \to \infty$, but

$\mathbb{F}_{K_n}(P_n) = \frac{2}{\sqrt{3}}\mathbb{F}(T) \not\to \mathbb{F}(T)$

* need more sophisticated tools...

Polyhedral Approximation

- Federer (1969) (4.2.21, 4.2.24)

* for normal current T in compact $K \subset \mathbb{R}^n$, and $\rho > 0$, $\exists$ normal polyhedral chain P s.t.

$$\mathbf{M}(P) < \mathbf{M}(T) + \rho,$$
$$\mathbf{M}(\partial P) < \mathbf{M}(\partial P) + \rho, \text{ and}$$
$$\mathbb{F}_K(T, P) < \rho.$$

POLYHEDRAL APPROXIMATION

- Federer (1969) (4.2.21, 4.2.24)

* For integral current T in compact $K \subset \mathbb{R}^n$, and $\rho > 0$, $\exists$ integral polyhedral chain P s.t.

$$\mathbf{M}(P) < \mathbf{M}(T) + \rho,$$
$$\mathbf{M}(\partial P) < \mathbf{M}(\partial P) + \rho, \text{ and}$$
$$\mathbb{F}_K(T, P) < \rho.$$

POLYHEDRAL APPROXIMATION

- Federer (1969) (4.2.21, 4.2.24)

✱ For integral current T in compact $K \subset \mathbb{R}^n$, and $\rho > 0$, $\exists$ integral polyhedral chain P s.t.

$$\mathbb{M}(P) < \mathbb{M}(T) + \rho,$$
$$\mathbb{M}(\partial P) < \mathbb{M}(\partial P) + \rho, \text{ and}$$
$$\mathbb{F}_K(T, P) < \rho.$$

- mass expansion $\to 0$ as $\rho \to 0$, but P need not be simplicial

SIMPLICIAL DEFORMATION

— Ibrahim, K, Vixie (2013)

* T can be deformed to chains $P, \partial P$ in a simplicial complex K s.t.

$$M(P) \leq c_1 M(T) + \Delta c_2 M(\partial T),$$
$$M(\partial P) \leq c_2 M(\partial T), \text{ and}$$
$$\mathbb{F}(T,P) \leq \Delta c_1 [M(T) + c_3 M(\partial T)].$$

$\Delta \to 0$, but

$c_1, c_2, c_3 > 1$

SIMPLICIAL DEFORMATION

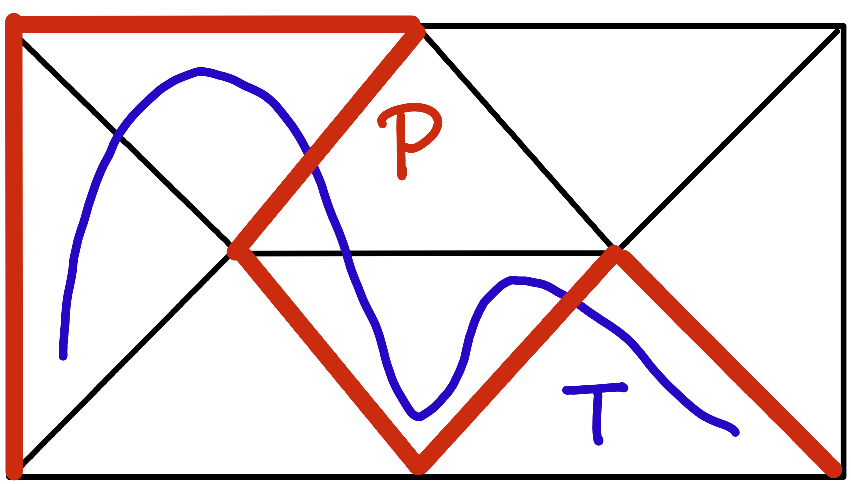
— Ibrahim, K, Vixie (2013)

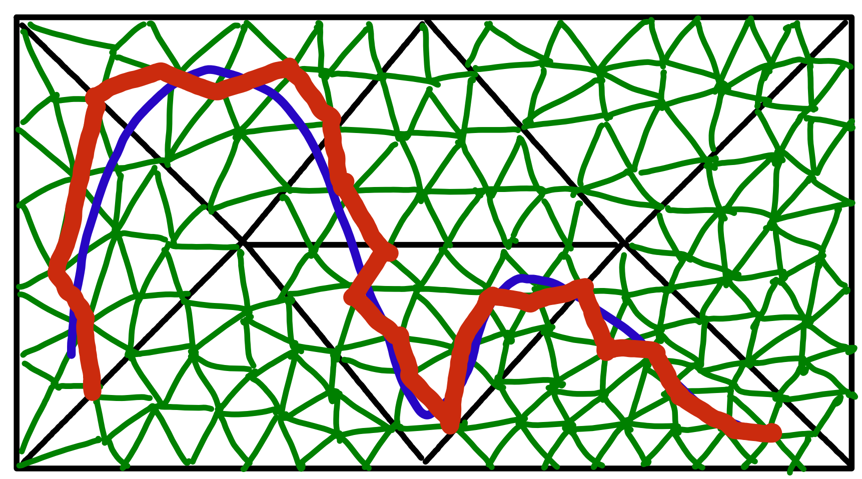
* T can be deformed to chains P, ∂P in a simplicial complex K s.t.

$M(P) \leq c_1 M(T) + \Delta\, c_2 M(\partial T)$, $\quad$ $\Delta \to 0$, but

$M(\partial P) \leq c_2 M(\partial T)$, and $\quad$ $c_1, c_2, c_3 > 1$

$\mathbb{F}(T,P) \leq \Delta c_1 [M(T) + c_3 M(\partial T)]$.





Simplicial Deformation

– Ibrahim, K, Vixie (2013)

* T can be deformed to chains $P, \partial P$ in a simplicial complex K s.t.

$$M(P) \leq c_1 M(T) + \Delta c_2 M(\partial T),$$
$$M(\partial P) \leq c_2 M(\partial T), \text{ and}$$
$$\mathbb{F}(T,P) \leq \Delta c_1 [M(T) + c_3 M(\partial T)].$$

$\Delta \to 0$, but

$c_1, c_2, c_3 > 1$

* (new) multi-current simplicial deformation theorem
  – simultaneously deform $T_1, \dots, T_m$ and $S_1, \dots, S_n$ onto K with similar bounds

Main Result: Overview

$$\begin{array}{c} T \\ \Big\downarrow{\scriptstyle \mathbb{F}} \\ X + \partial S \end{array}$$

Main Result: Overview

$$
\begin{tikzcd}
T \arrow[r, "\text{polyh. aprx}"] \arrow[d, "\mathbb{F}"'] & P_\delta \\
X + \partial S \arrow[r, "\text{polyh. aprx}", "\substack{X \to X_\delta \\ S \to S_\delta}"'] & X_\delta + \partial S_\delta
\end{tikzcd}
$$

Main Result: Overview

$$\begin{array}{ccc} T & \xrightarrow{\text{polyh. aprx}} & P_\delta \\ \Big\downarrow \mathbb{F} & & \\ X + \partial S & \xrightarrow[\substack{X \to X_\delta \\ S \to S_\delta}]{\text{polyh. aprx}} & X_\delta + \partial S_\delta \end{array}$$

$P_\delta \neq X_\delta + \partial S_\delta$, but difference is probably small

MAIN RESULT: OVERVIEW

$$
\begin{tikzcd}
T \arrow[r, "\text{polyh. aprx}"] \arrow[d, "\mathbb{F}"'] & P_\delta \\
X + \partial S \arrow[r, "\text{polyh. aprx}", "{\substack{X \to X_\delta \\ S \to S_\delta}}"'] & X_\delta + \partial S_\delta
\end{tikzcd}
$$

$P_\delta \neq X_\delta + \partial S_\delta$, but difference is probably small

* find simplicial complex K_δ triangulating $P_\delta, X_\delta, S_\delta$ with mass expansion L indep. of δ (in simplicial deformation theorem)

Main Result: Overview

* show $\mathbb{F}(T) = \lim_{\delta \downarrow 0} \mathbb{F}_{K_\delta}(P_\delta)$, i.e.,

  simplicial flat norm $\approx$ continuous flat norm

MAIN RESULT: OVERVIEW

- show $\mathbb{F}(T) = \lim_{\delta \downarrow 0} \mathbb{F}_{K_\delta}(P_\delta)$, i.e., simplicial flat norm $\approx$ continuous flat norm
- P_δ has integral SFN decomposition $\Longrightarrow$ so does T under flat norm

MAIN RESULT: OVERVIEW

* show $\mathbb{F}(T) = \lim_{\delta \downarrow 0} \mathbb{F}_{K_\delta}(P_\delta)$, i.e.,
  Simplicial flat norm $\approx$ continuous flat norm
* P_δ has integral SFN decomposition $\Rightarrow$
  so does T under flat norm
* K_δ? Conjecture: There exists subdivision of K_δ that is regular in most places, with irregularity pushed to "corners of small mass".

MAIN RESULT: OVERVIEW

* show $\mathbb{F}(T) = \lim_{\delta \downarrow 0} \mathbb{F}_{K_\delta}(P_\delta)$, i.e.,
  Simplicial flat norm $\approx$ continuous flat norm
* P_δ has integral SFN decomposition $\Rightarrow$
  so does T under flat norm
* K_δ? Conjecture: There exists subdivision of K_δ that is regular in most places, with irregularity pushed to "corners of small mass".

✓ true in 2D (Shewchuk 2002).

OPEN QUESTIONS

? Extending result to higher dimensions?
- our framework works in any dimension, assuming subdivision conjecture holds

OPEN QUESTIONS

? Extending result to higher dimensions?
- our framework works in any dimension, assuming subdivision conjecture holds

? Other regularization methods?

OPEN QUESTIONS

? Extending result to higher dimensions?
- our framework works in any dimension, assuming subdivision conjecture holds

? Other regularization methods?

? Codimension 2?

OPEN QUESTIONS

? Extending result to higher dimensions?
- our framework works in any dimension, assuming subdivision conjecture holds

? Other regularization methods?

? Codimension 2?

? Other questions where discrete $\Rightarrow$ continuous?