The Maximum Distance Problem and Minimum Spanning Trees

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The Analyst's Traveling Salesman Thm.

Under what conditions can a subset E of $\mathbb{R}^n$ be covered by a rectifiable curve?

1. In $\mathbb{R}^2$: Jones, Peter (1990). "Rectifiable sets and the Traveling Salesman Problem". Inventiones Mathematicae. 102: 1–15.
2. In $\mathbb{R}^n$: Okikiolu, Kate (1992). "Characterization of subsets of rectifiable curves in Rn". Journal of the London Mathematical Society. 46 (2): 336–348.

Both papers used what are now called *Jones' beta numbers* and gave *quantitative* conditions on E such that it may be covered by a rectifiable curve.

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Beta Numbers

Let $E \subset \mathbb{R}^2$ and let Q be a cube. The Beta number of E within the cube Q is given by

$$\beta_E(Q) := \inf_L \sup_{y \in E \cap Q} \frac{\mathrm{dist}_\infty(y, L)}{l(Q)}.$$

Consider the sum of $\beta_E(Q)$ over the family of dyadic cubes Δ:

$$\beta(E) := \mathrm{diam}\, E + \sum_{Q \in \Delta} \beta_E(Q)^2 l(Q).$$

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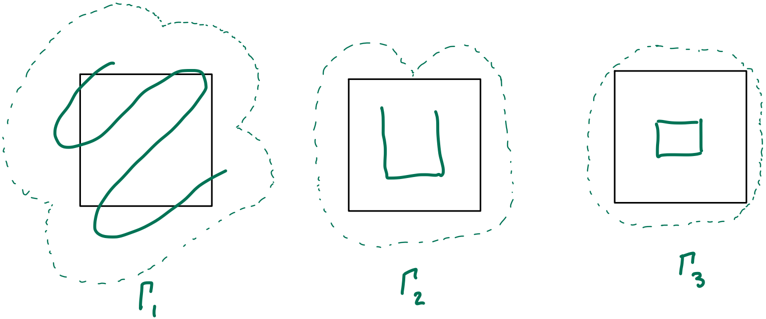
Theorem

There exists a number C such that if $\beta(E) < \infty$ then $E \subset \mathbb{R}^2$ can be covered by a rectifiable curve of length no more than $C\beta(E)$. Conversely, if E can be covered by a rectifiable curve Γ, then $\beta(E) < C\mathcal{H}^1(\Gamma)$.

The Maximum Distance Problem

Although, for example, a cube cannot be covered by a rectifiable curve, for any $\epsilon > 0$ there are many curves whose ϵ-neighborhoods $B(\Gamma, \epsilon)$ can cover E.



The Maximum Distance Problem

Let $E \subset \mathbb{R}^n$ be bounded and $\epsilon > 0$. Find the minimizers of

$$\lambda(E, \epsilon) := \inf\{\mathcal{H}^1(\Gamma) : \Gamma \text{ is compact, connected and } B(\Gamma, \epsilon) \supset E\}.$$

A Naive Heuristic

1. Cover E with some number of closed ϵ balls with centers X.
2. Find a Steiner tree S_X spanning X.

Main Question

How close is $\mathcal{H}^1(S_X)$ to $\lambda(E, \epsilon)$?

(ϵ, k)-minimal Steiner length

Let $E \subset \mathbb{R}^n$ be compact and $\epsilon > 0$, we define the (ϵ, k)-minimal Steiner length of E to be

$$\sigma_k(E, \delta) := \inf\{\mathcal{H}^1(S_X) : X \in \mathcal{N}_k(E, \epsilon),$$

where $\mathcal{N}_k(E, \epsilon)$ is the family of all k-point sets whose ϵ-balls cover E and S_X is any Steiner tree over X.

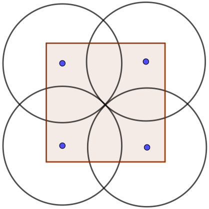


Figure: $X_1 \in \mathcal{N}_4(E, \epsilon)$

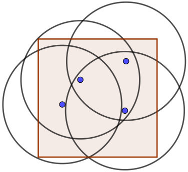


Figure: $X_2 \in \mathcal{N}_4(E, \epsilon)$

(ϵ, k)-minimal Steiner length

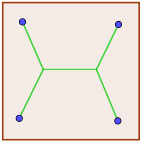


Figure: S_{X_1} for $X_1 \in \mathcal{N}_4(E, \epsilon)$.

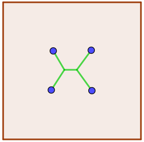
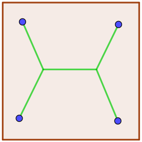


Figure: $(\epsilon, 4)$-minimal Steiner tree

(ϵ, k)-minimal Steiner length



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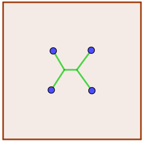

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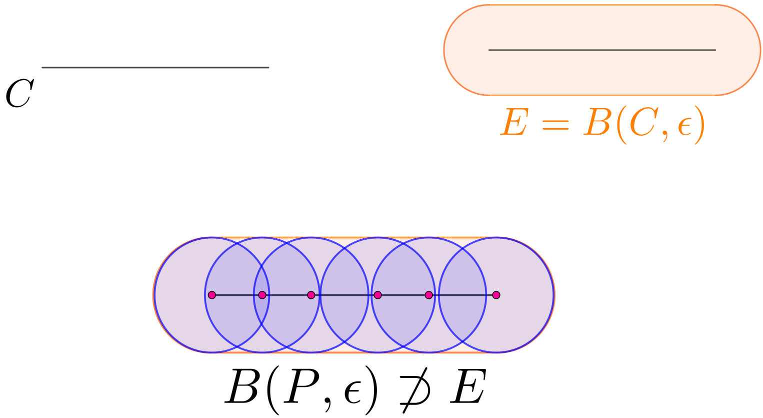
How close are $\sigma_k(E,\delta)$ and $\lambda(E,\epsilon)$?

1. $\sigma_k(E,\epsilon) \neq \lambda(E,\epsilon)$ for any $k \in \mathbb{N}$.



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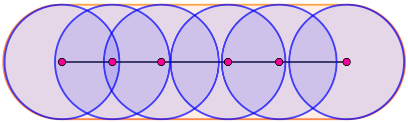

$$B(P,\epsilon) \not\supset E$$

How close are $\sigma_k(E, \delta)$ and $\lambda(E, \epsilon)$?

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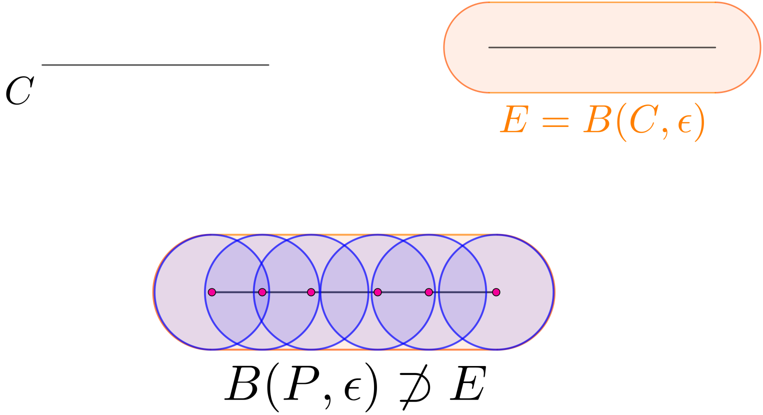

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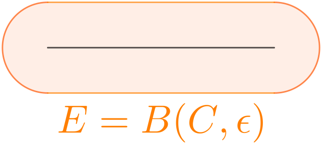
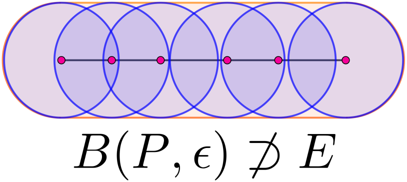




Modified question

What if we allow an arbitrary number of points? i.e.

$$\sigma(E,\epsilon) := \inf\{\mathcal{H}^1(S_X) : X \in \mathcal{N}(E,\epsilon) \quad where \quad \mathcal{N}(E,\epsilon) = \cup_{i=1}^{\infty}\mathcal{N}_k(E,\epsilon)\}$$

The Case of $E = B(L, \epsilon)$

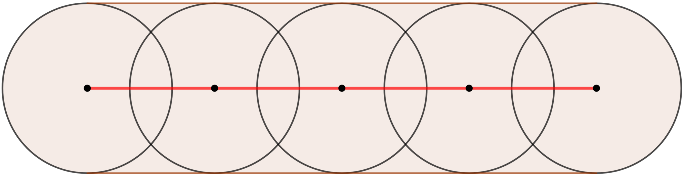


Figure: Partition L into N subintervals of length L/N.

The Case of $E = B(L, \epsilon)$

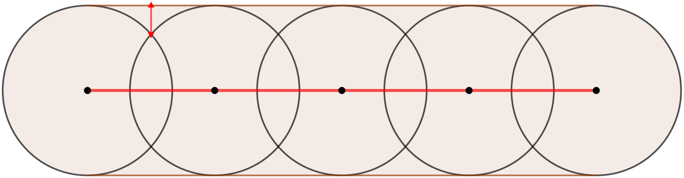


Figure: We must lift each circle up and down by $\delta := \delta(L, N, \epsilon)$.

The Case of $E = B(L, \epsilon)$

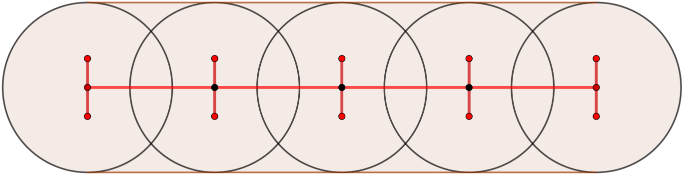


Figure: Construct a polygonal curve C_N of length $(N+1)2\delta + L$ spanning points X_N whose ϵ balls cover E.

The Case of $E = B(L, \epsilon)$

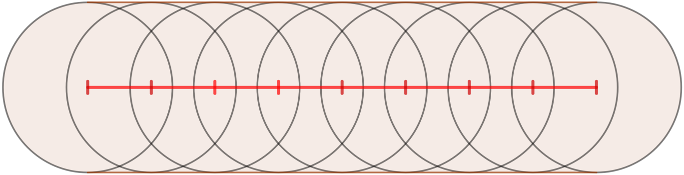
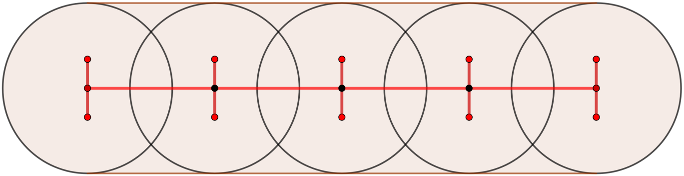


Figure: Increase N and see that δ is smaller.

The Case of $E = B(L, \epsilon)$



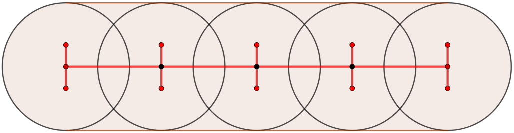
- We have that

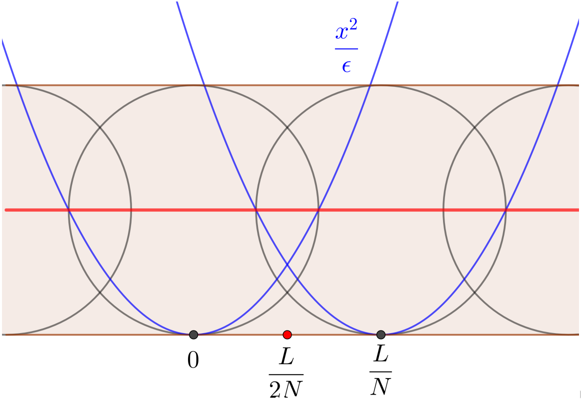
$$L \leq \mathcal{H}^1(S_{X_N}) \leq \mathcal{H}^1(C_N)$$
$$= (N+1)2\delta + L.$$

- If we can show that $\mathcal{H}^1(C_N) \to L$ as $N \to 0$ we get

$$\sigma(E, \epsilon) = L = \lambda(E, \epsilon).$$

The Case of $E = B(L, \epsilon)$





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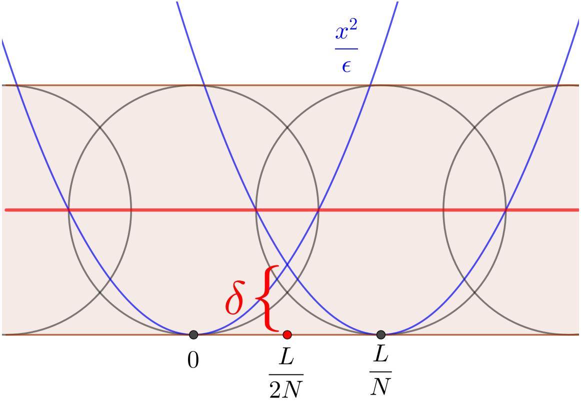


$$\delta = \frac{(\frac{L}{2N})^2}{\epsilon} = CN^{-2}$$

The Case of $E = B(L, \epsilon)$

We then have

$$\delta = \frac{(\frac{L}{2N})^2}{\epsilon} = CN^{-2}$$

implying

$$\begin{aligned} \mathcal{H}^1(C_N) &= (N+1)2\delta + L \\ &= C\frac{(N+1)}{N^2} + L \\ &\to L \qquad \text{as } N \to \infty. \end{aligned}$$

$\therefore$ For $E = B(L, \epsilon)$ we can instead minimize over Steiner trees which span points whose ϵ balls cover E.

The General Case in $\mathbb{R}^2$

Lemma 3.5 (A, Krishnamoorthy, Vixie; arXiv:2004.07323)

Let $\epsilon > 0$ and let $\Gamma \subset \mathbb{R}^2$ be a rectifiable curve. For any $\delta > 0$, there exists a finite point set $X = \{x_i\}_{i=1}^N$ and another rectifiable curve Γ_* containing X such that

$$B(X, \epsilon) \supset B(\Gamma, \epsilon) \qquad \text{and} \qquad \mathcal{H}^1(\Gamma_*) \leq \mathcal{H}^1(\Gamma) + \delta.$$

Theorem 3.7 (Miranda Jr., Paolini, Stepanov; Calc. Var. 27, 287–309 (2006))

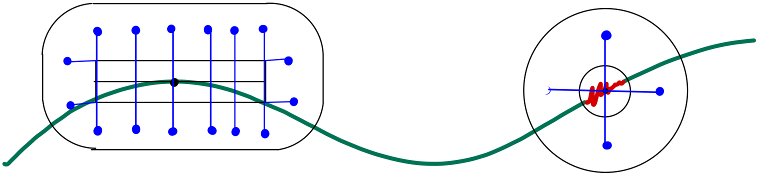
Let $\Gamma \subset \mathbb{R}^2$ be a rectifiable curve and with $F_E(\Gamma) > 0$. Then for each $\delta > 0$ there exists a compact connected Γ_* such that

$$F_E(\Gamma_*) < F_E(\Gamma) \qquad \text{and} \qquad \mathcal{H}^1(\Gamma_*) \leq \mathcal{H}^1(\Gamma) + \delta.$$

The General Case in $\mathbb{R}^2$

Proof of Lemma.

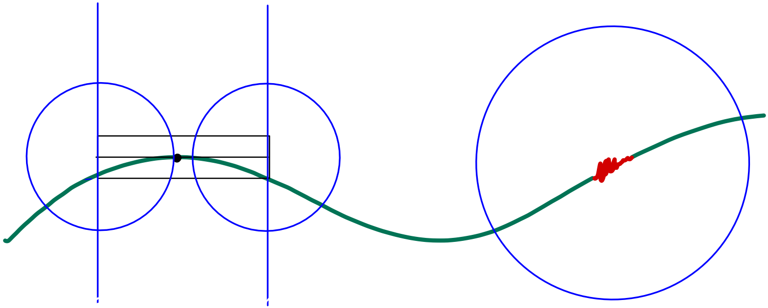
1. Parameterize Γ with a Lipschitz map $\gamma : I \to \mathbb{R}^2$ s.t. $\mathrm{L}(\gamma) \leq 2\mathcal{H}^1(\Gamma)$.
2. Partition the domain I s.t. a large fraction of I is made up of good pieces (G_i) and a small fraction of bad pieces (B_j).
3. Cover the neighborhoods of the $\gamma(G_i)$ with endpoints of prongs (as in the case of a line segment) and cover the neighborhoods of $\gamma(B_j)$ with endpoints of spokes.



The General Case in $\mathbb{R}^2$

Proof of Theorem.

1. Work only in the image of γ.
2. Use Egorof's theorem to get uniform estimates on a Beta number to decompose Γ into a good pieces and a bad piece.
3. Construct vertical lines and small circles around good pieces and construct larger circles around the bad pieces.



Future Work

- $\sigma_k(E, \epsilon)$ for fixed $k \in \mathbb{N}$ computationally and theoretically.
- Configuration spaces of $\mathcal{N}_k(E, \epsilon)$ and how they change as a function of k and ϵ.
- Minimal spanning trees over centers of random covers of E.

Thank You!