Washington State University

Department of Mathematics

Lattice-based Algorithms for Number Partitioning in the Hard Phase

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joint work with

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Department of Mathematics, WSU

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- allocate $\beta = 1/2\ \alpha$, or, as close as possible to β, to each subset

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- *balanced* NPP (BalNPP): $|S_1| = |S_2| = n/2$ (for even n)

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- $\triangle^* = 0$ (or $\triangle^* = 1$ when α odd) gives a *perfect* partition

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- theoretical

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  - phase transition (fully characterized mathematically)
  - NP-completeness of other problems involving numbers – bin packing, knapsack etc.

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    - Karmarkar, Karp, Lueker, Odlyzko (88): median $\triangle^*$ for NPP
    - Lueker (98): average $\triangle^*$ for NPP
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- minimum partition unique for $R \gg 2^n$

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- running time is $O(n \log n)$

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- also consider replacing two largest numbers by their *sum*
- improves on KK discrepancy as it continues to run
- effective in practice in the easy phase
- $\#$ of branch-and-bound nodes is exponential in n when $R > 2^n$
- converges very slowly

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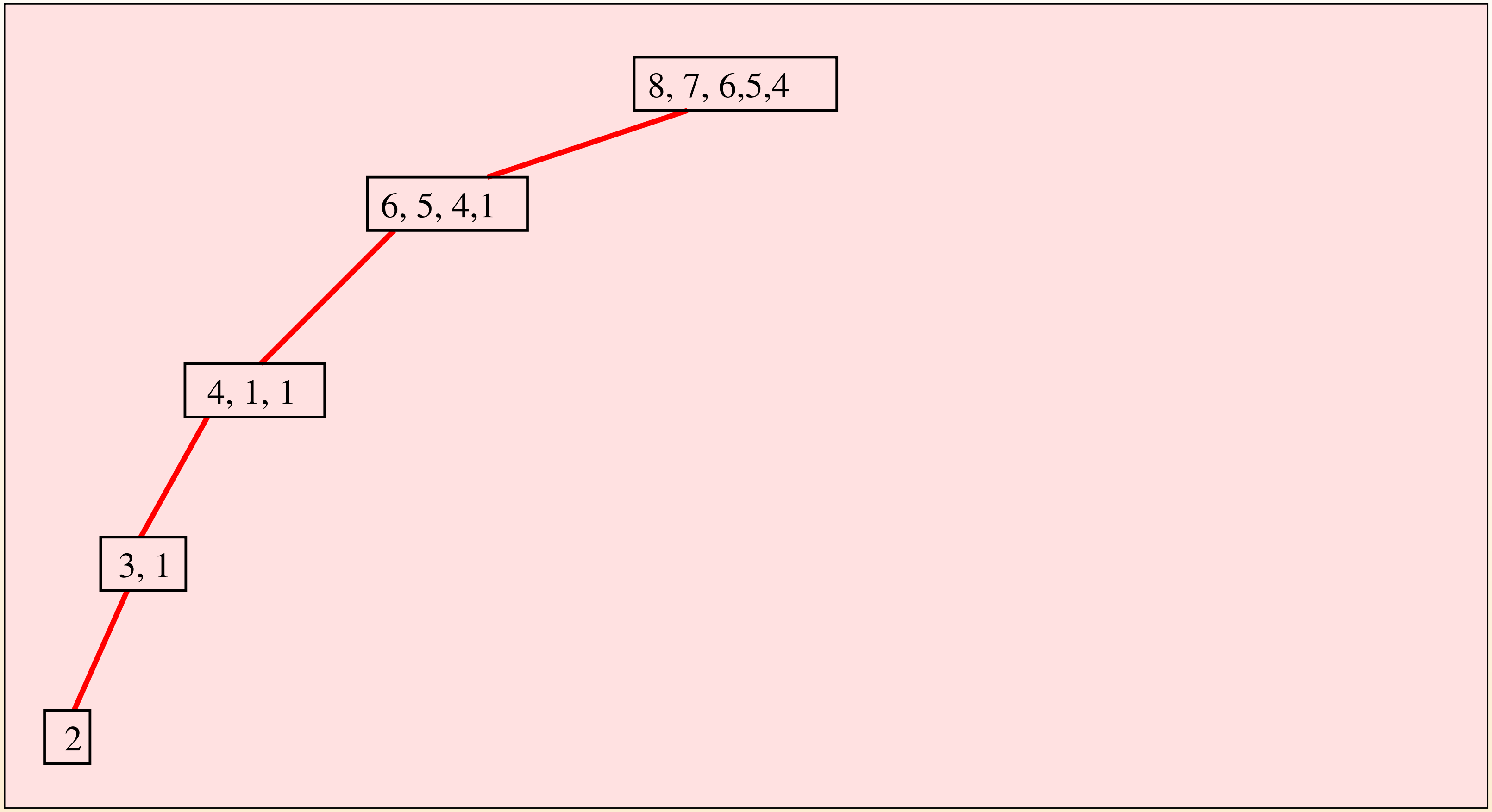

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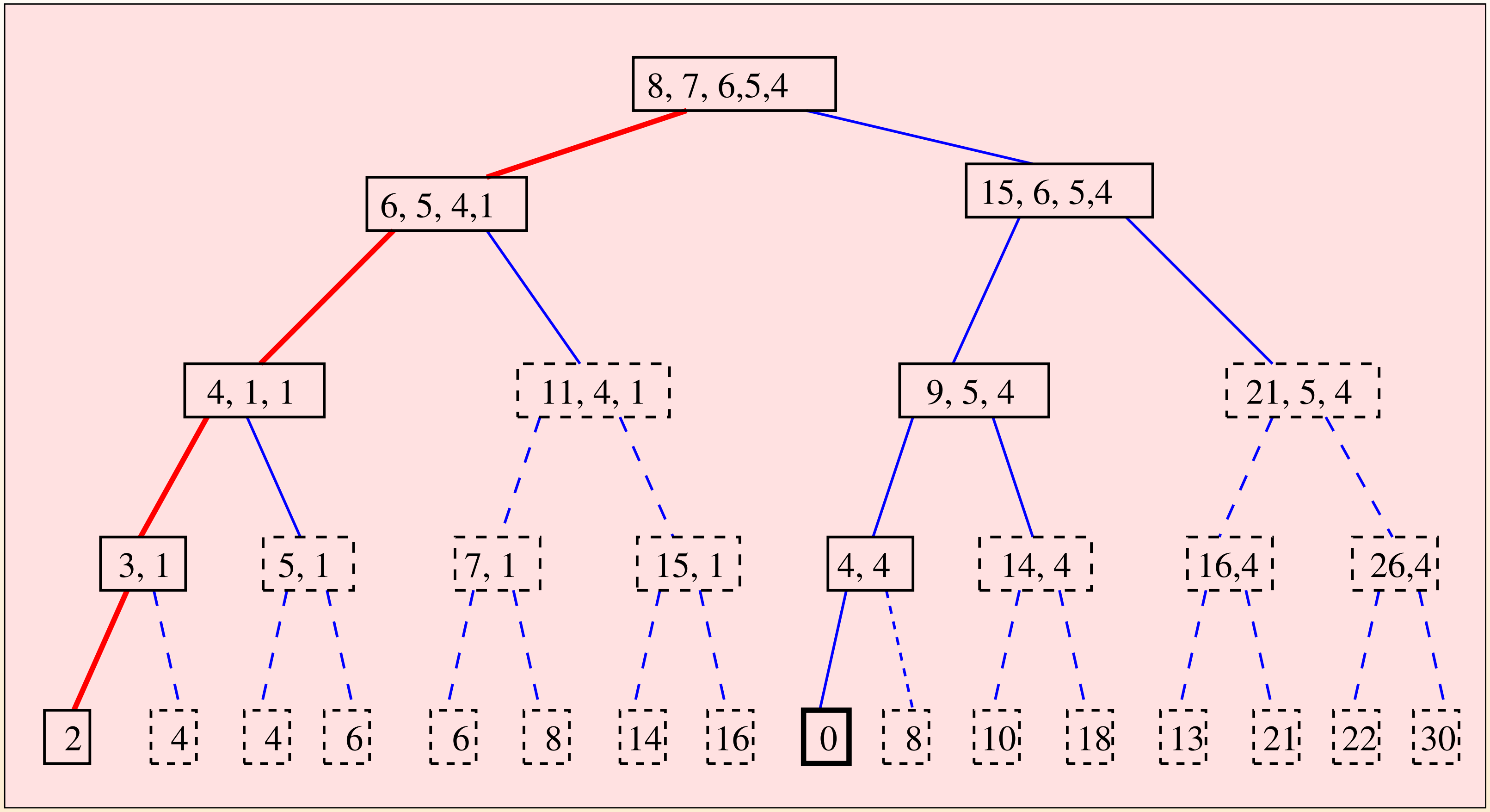

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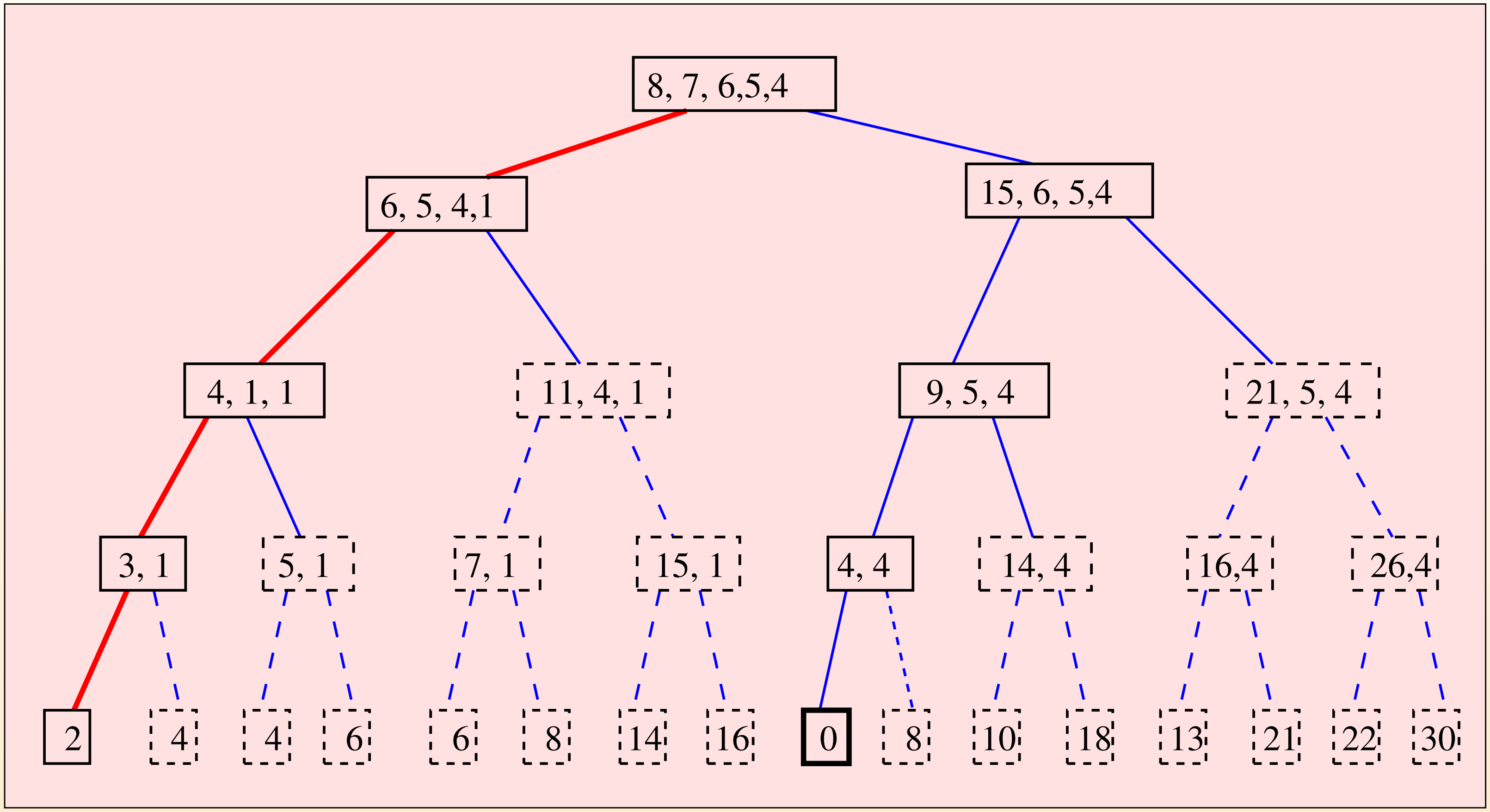

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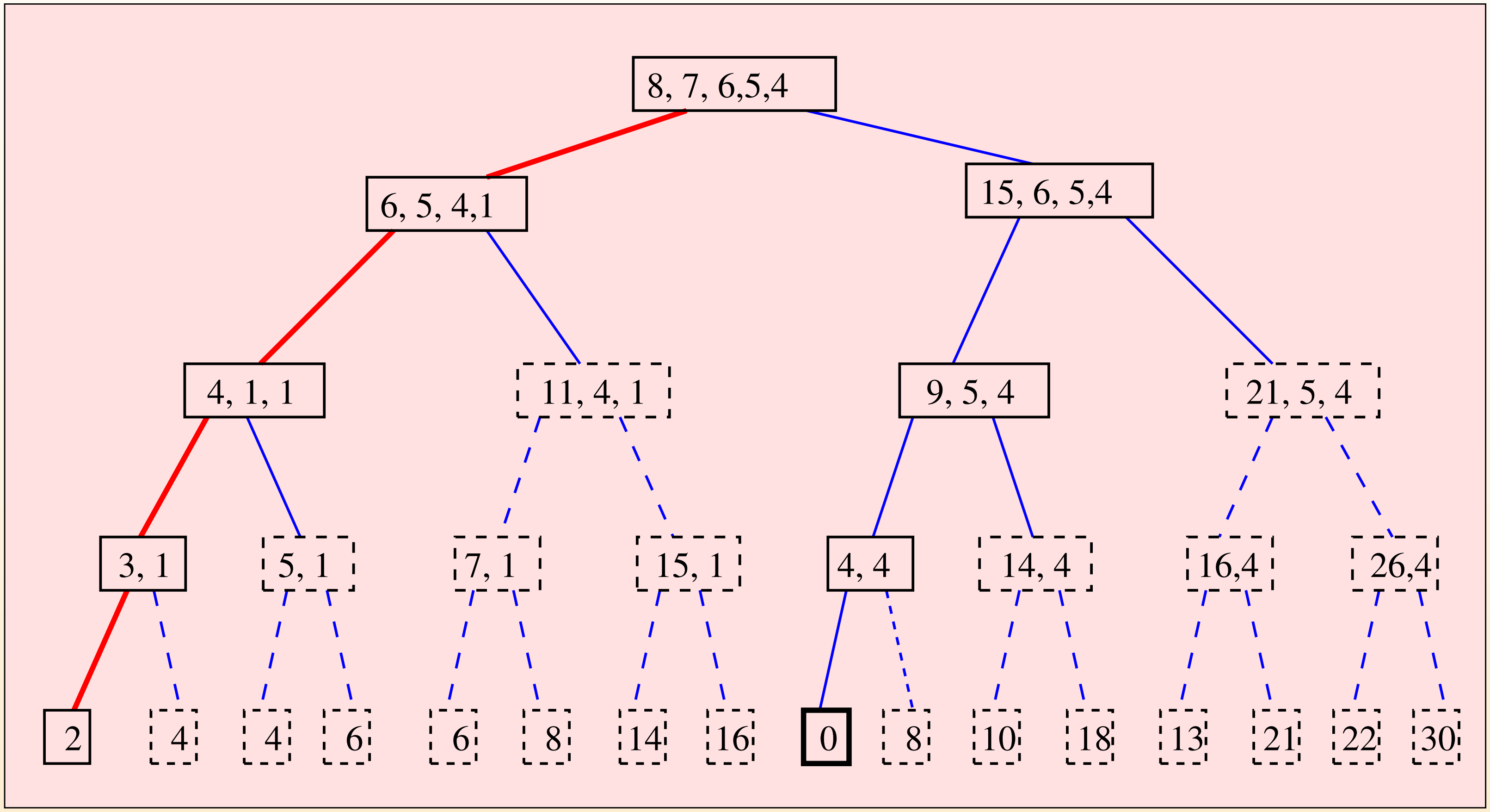

- dashed parts of the tree are pruned
- two-color associated tree to recover partition

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- typical numbers are *huge*; for $n = 30$, look at a_j's with 11 digits!

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- output of reduction: DCVP instance $(B, \boldsymbol{u}, d\sqrt{n+1})$
- generalization of Micciancio (2001) reduction of subset sum to CVP

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Theorem 2. *DBalNPP_d is Cook-reducible to DCVP for $d > 0$.*

$$B' = \begin{bmatrix} 2d\, I \\ (d+1)\mathbf{1}^T \\ \boldsymbol{a}^T \end{bmatrix}, \quad \boldsymbol{u'} = \begin{bmatrix} d\,\mathbf{1} \\ (d+1)\lfloor n/2 \rfloor \\ \beta \end{bmatrix}.$$

DBALNPP to DCVP

- DBALNPP_d: Given $a_1, \ldots, a_n$ and an even number $2d > 0$, decide if a balanced partition exists with $\triangle \leq 2d$. Equivalently, find $\boldsymbol{x} \in \{0,1\}^n$ with $\sum_j x_j = \lfloor n/2 \rfloor$ s.t. $\sum_j a_j x_j = \beta - \delta$ or $\sum_j a_j x_j = \beta + \delta$, for some $\delta \leq d$, if it exists.

  Theorem 2. DBALNPP_d *is Cook-reducible to* DCVP *for* $d > 0$.

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- output of reduction: DCVP instance $(B', \boldsymbol{u'}, d\sqrt{n+1})$

A lattice algorithm for NPP

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- Lagarias & Odlyzko (85), Coster et al. (92): for subset sums

BR Algo Tests:

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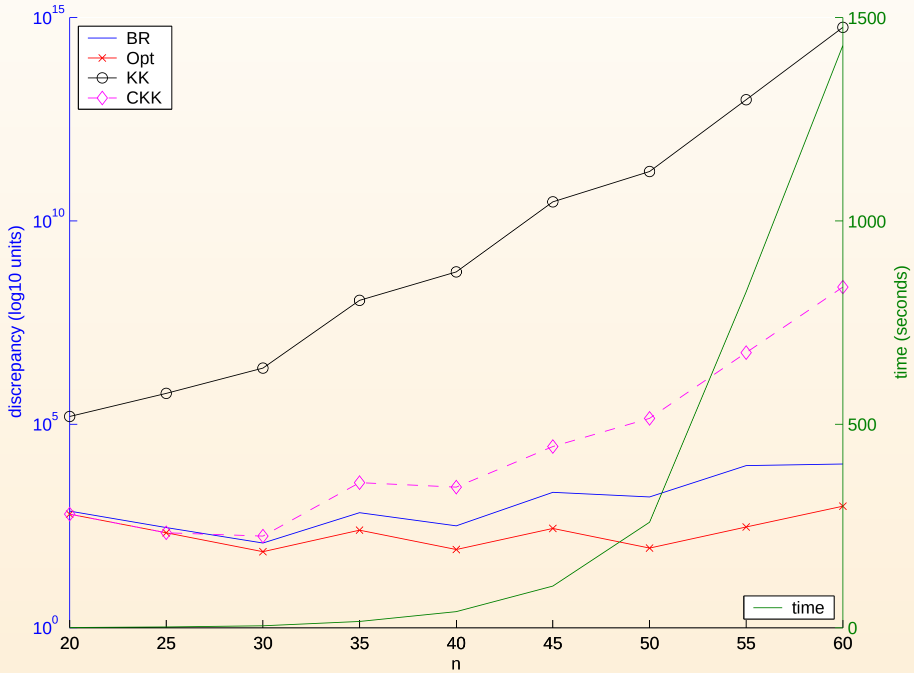

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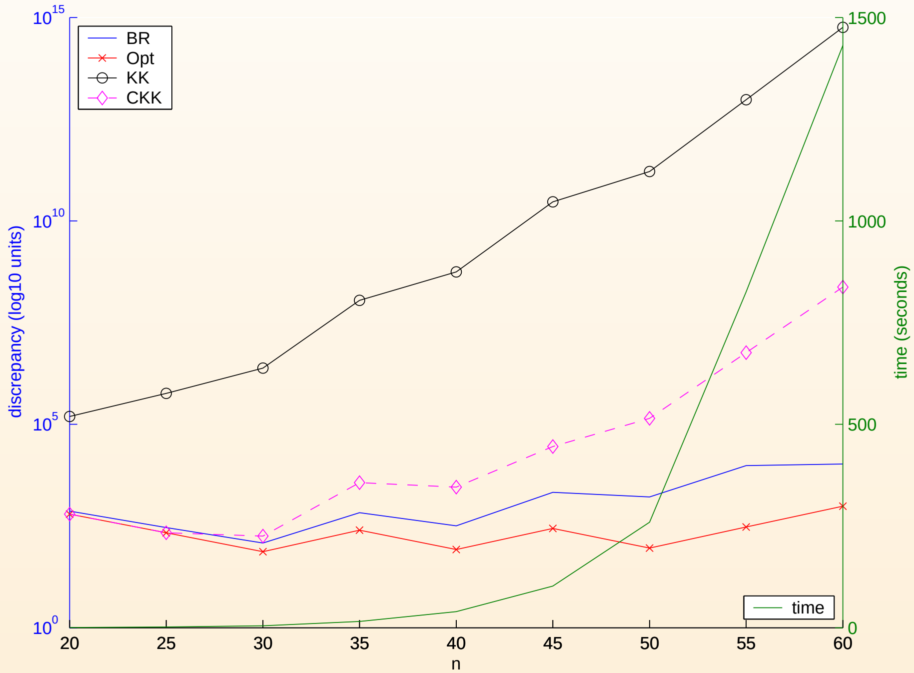

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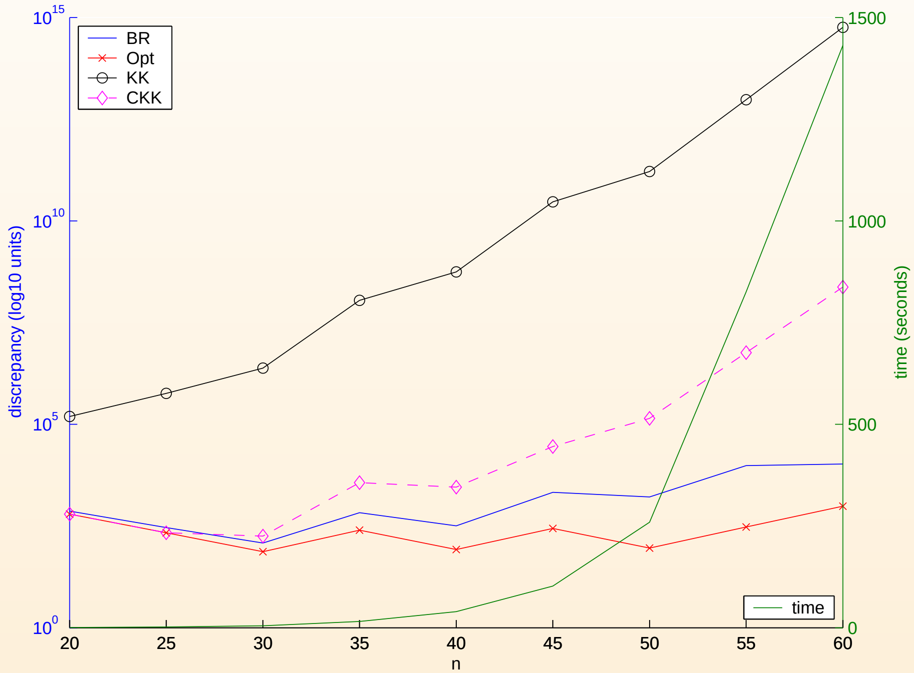

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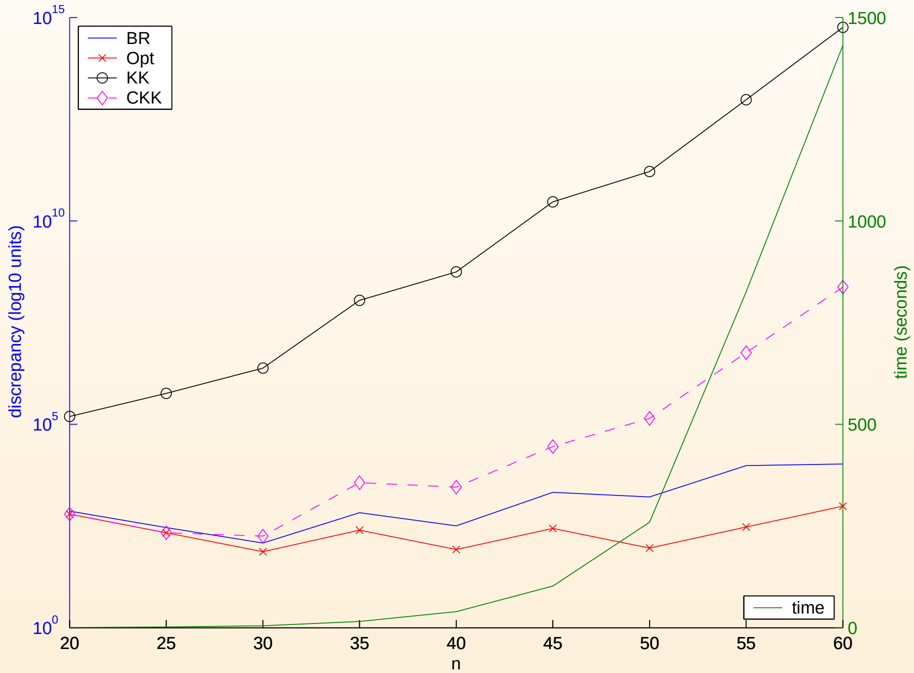

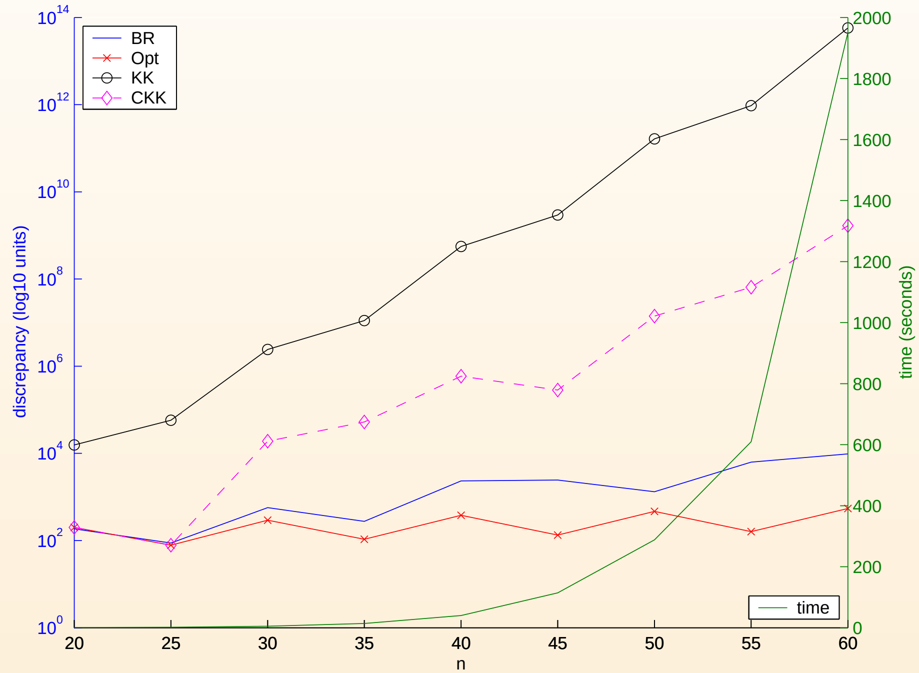

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  MIP for NPP:

$$
\begin{array}{llll}
\min & 2w & & \\
\text{s.t.} & w & \geq & \sum a_j x_j - \beta \\
& w & \geq & -\sum a_j x_j + \beta \\
& x_j & \in & \{0,1\} \qquad\qquad j = 1, \ldots, n.
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- solve the *rangespace reformulation* using standard solver:

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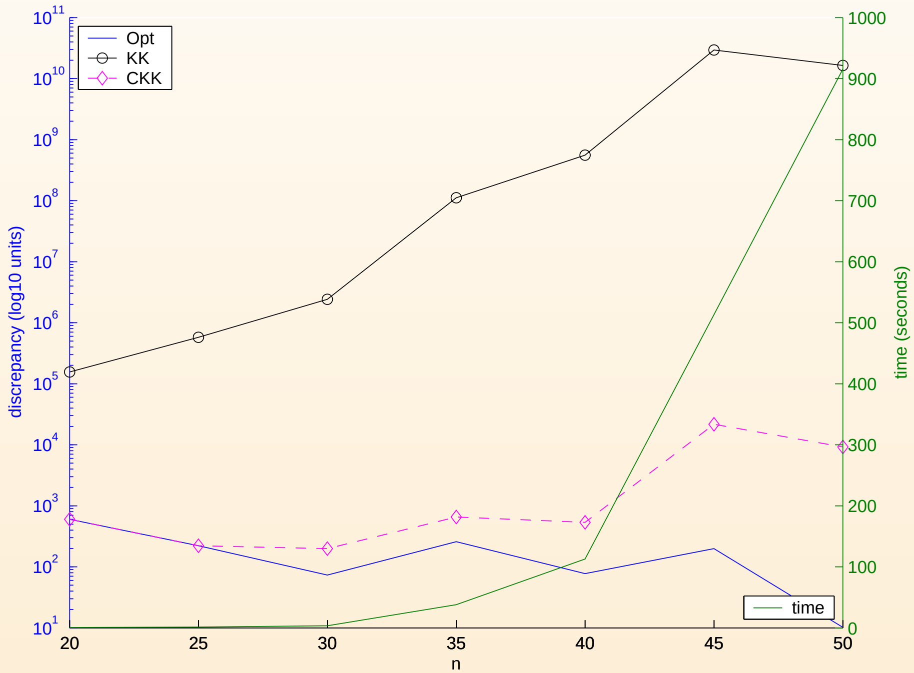
RSRef Tests:

RSRef Tests: on NPP



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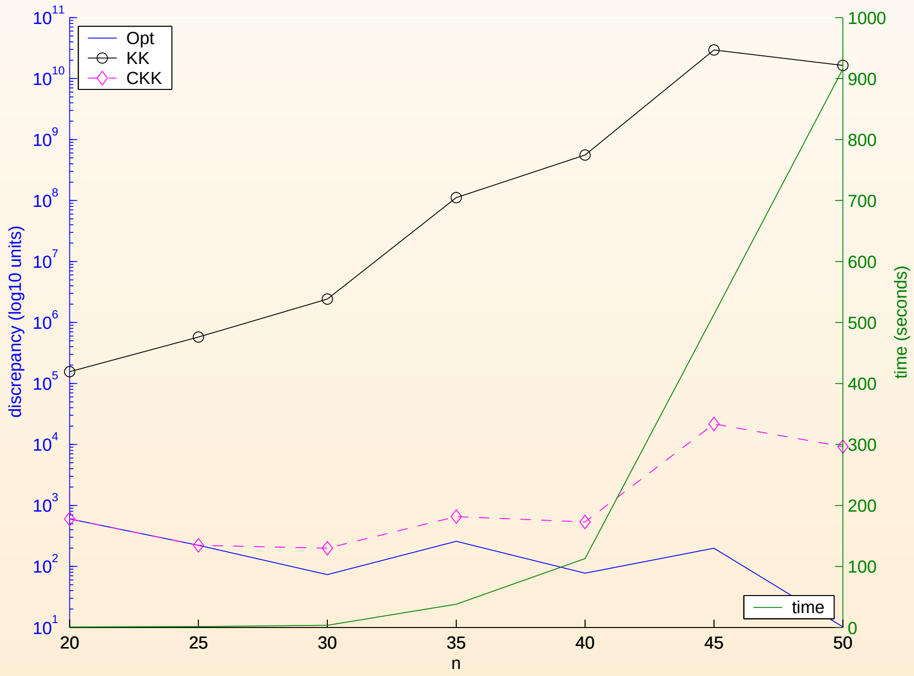

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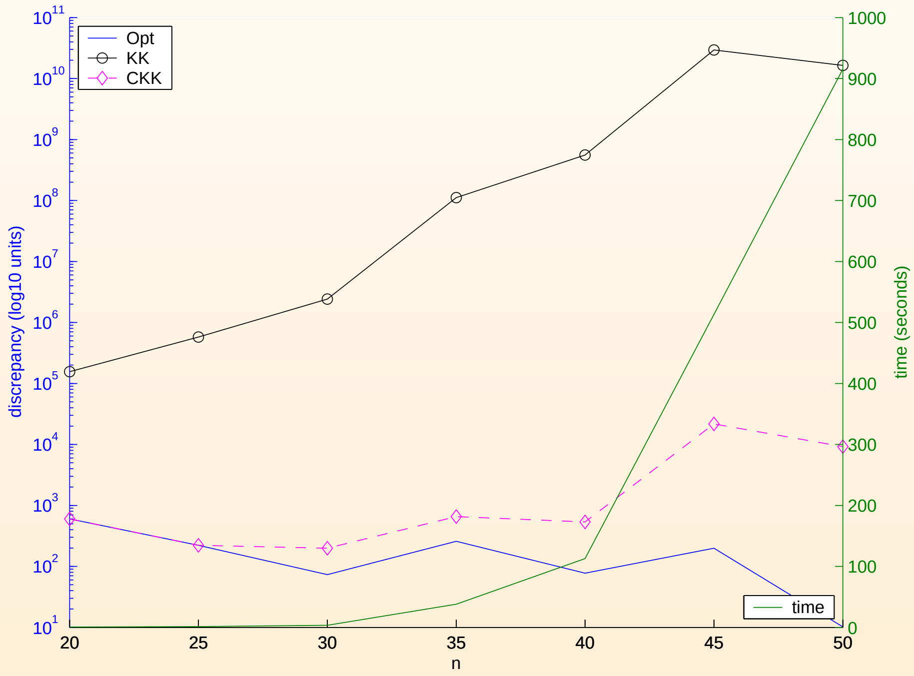
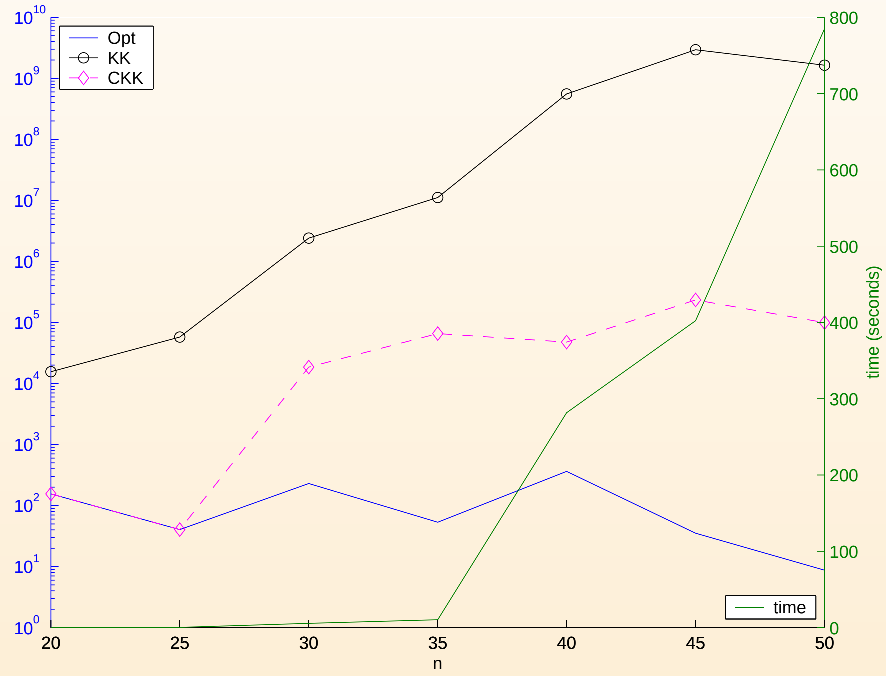
RSRef Tests: on NPP and BalNPP





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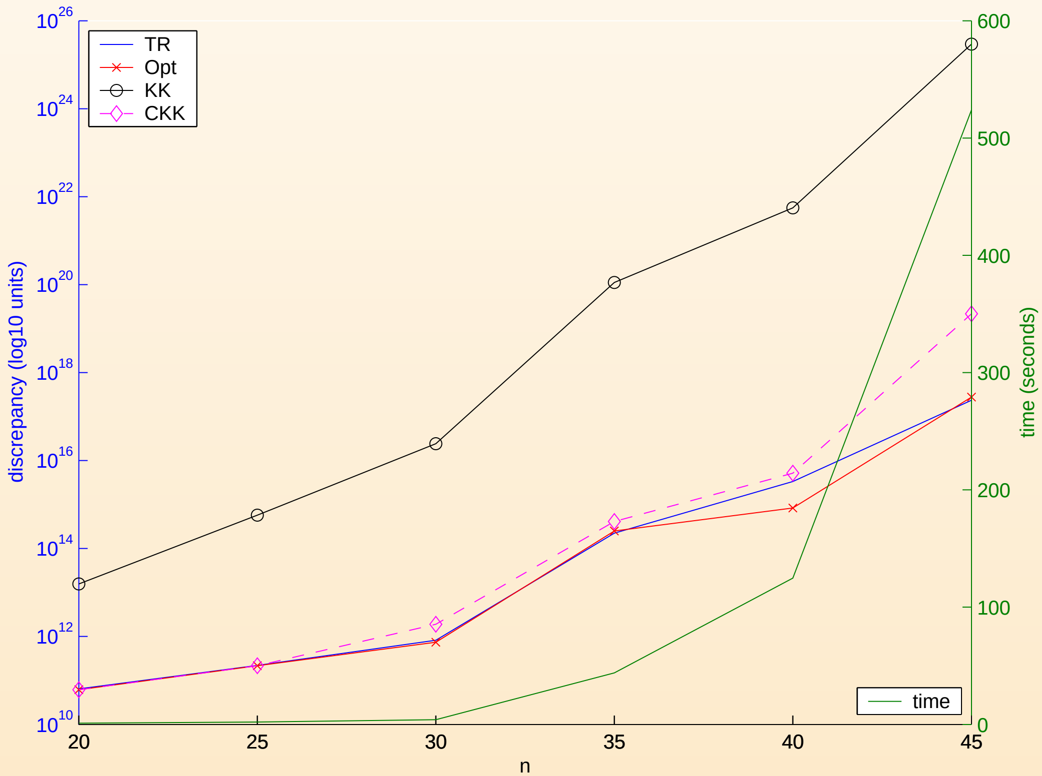

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Outline

- Number Partitioning Problem (NPP)
- Karmarkar-Karp differencing (KK)
- NPP and the Closest Vector Problem (CVP)
- A Basis Reduction Heuristic for NPP
- Mixed Integer Program (MIP) for NPP
- Truncated NPP