Lattices and Integer Optimization - A Tutorial (Part II)

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Basis Reduction (BR) and Discrete Optimization

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  - simple; works in practice
  - theory for class of knapsack problems,
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- lattice-based approaches to number partitioning in hard phase (joint work with Bill Webb, Nathan Moyer (WSU))

Integer Programming (IP)

- IP feasibility

  Given

$$P \; = \; \{ \mathbf{x} \,|\, \boldsymbol{\ell} \leq A\mathbf{x} \leq \mathbf{b} \},$$

  Find $\mathbf{x} \in P \cap \mathbb{Z}^n$, or prove that no such $\mathbf{x}$ exists.

Branching for IP feasibility

Given polyhedron P, integral vector $\mathbf{c}$,

- $\mathrm{width}(\mathbf{c}, P) = \max\{\, \mathbf{c}^T\mathbf{x} \mid \mathbf{x} \in P \,\} - \min\{\, \mathbf{c}^T\mathbf{x} \mid \mathbf{x} \in P \,\} = \gamma - \delta.$

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- when $\mathbf{c} = \mathbf{e}_i$, we are branching on single variable x_i
- different choices of $\mathbf{c}$ produce very different effects on branching

Column-BR in rangespace (CBR-R)

$$\begin{aligned} P &= \{\, \mathbf{x} \,|\, \boldsymbol{\ell} \leq \; A\mathbf{x} \leq \mathbf{b} \,\} \\ \tilde{P} &= \{\, \mathbf{y} \,|\, \boldsymbol{\ell} \leq (AU)\mathbf{y} \leq \mathbf{b}\} \end{aligned}$$

where U is unimodular

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- reformulation using Hermite Normal Form (HNF) computation

$$\{\, \mathbf{x} \in \mathbb{Z}^n \,|\, A_1\mathbf{x} = \mathbf{b}_1 \,\} \;=\; \{\, \mathbf{x}_d + B_1\boldsymbol{\lambda} \,|\, \boldsymbol{\lambda} \in \mathbb{Z}^{n-m} \,\}$$

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- a "dual" method

CBR-N v/s CBR-R

- numerical output of CBR-N is similar to the reformulation technique of Aardal, Hurkens, and Lenstra (1998) – going from n vars, m equations to $n - m$ vars, no equations

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- both CBR-R and CBR-N actually work for essentially all hard IPs used to test "non-traditional" IP algorithms

$t+1$-level decomposable knapsack problems

- $\mathbf{a} = \mathbf{p}_1 M_1 + \mathbf{p}_2 M_2 + \cdots + \mathbf{p}_t M_t + \mathbf{r}$, with $M_1 > M_2 > \cdots > M_t$ and for suitable β, δ

$$(KP) \quad \beta \leq \mathbf{a}\,\mathbf{x} \;\leq \beta + \delta, \qquad \mathbf{0} \leq \mathbf{x} \leq \mathbf{u}, \quad \mathbf{x} \in \mathbb{Z}^n$$

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- if $t = 1$, we just write $\mathbf{p} = \mathbf{p}_1$, $M = M_1$

DKP instances for $t = 1$

- a classic example: Jeroslow's problem

$$\begin{array}{rcl} 2(x_1 + \cdots + x_n) & = & n \\ x_i \in \{0, 1\}^n & & \end{array}$$

  where n is odd.

- with B&B branching on the x_i, no node is pruned above level $n/2$
- if we branch on $x_1 + \cdots + x_n$, we solve it at the root node
- here $\mathbf{p} = \mathbf{1}$, $\mathbf{r} = \mathbf{0}$, $M = 2$

Other known instances for $t = 1$

- $\mathbf{p} = \mathbf{1}$, $\mathbf{r} = (2^0, \ldots, 2^{n-1})$, $\mathbf{u} = \mathbf{1}$, $M = 2^{n+\ell+1}$: Todd's problem from Chvátal "Hard knapsack problems" (1980)

- $\mathbf{p} = \mathbf{1}$, $\mathbf{r} = (1, \ldots, n)$, $\mathbf{u} = \mathbf{1}$, $M = n(n+1)$: Avis' problem from same paper

- Gu, Nemhauser, Savelsbergh (2001) - modification of Todd's problem

- Cornuéjols, Urbaniak, Weismantel, Wolsey (1996): $\mathbf{p} > \mathbf{0}$, $\mathbf{u} = +\infty$ (inequality)

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All, except the last two take $\geq 2^{n/2}$ nodes for ordinary B&B.
In last, $\exists$ a large rhs for which the problem is infeasible

Recipe for DKPs and hardness

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- lower bound on the number of nodes necessary for *ordinary* B&B (using x_j's)

DKPs get harder as t grows

two infeasible knapsack problems: can you tell which one is harder?

$$\begin{aligned} 1473x_1 + 1524x_2 + 1569x_3 + 1570x_4 + 1575x_5 + 1624x_6 + 1625x_7 \\ +2160x_8 + 2206x_9 + 2207x_{10} + 2211x_{11} + 2211x_{12} + 2257x_{13} \\ +2260x_{14} + 2305x_{15} + 2843x_{16} + 2943x_{17} + 2947x_{18} + 2991x_{19} \\ +2993x_{20} + 2997x_{21} + 3528x_{22} + 3577x_{23} + 3631x_{24} + 3677x_{25} \\ = 28980,\ x_i \in \{0, 1\}. \end{aligned}$$

$$\begin{aligned} 1314x_1 + 1315x_2 + 1317x_3 + 1318x_4 + 1971x_5 + 1972x_6 + 1973x_7 \\ +1976x_8 + 1977x_9 + 1977x_{10} + 2629x_{11} + 2630x_{12} + 2631x_{13} \\ +2631x_{14} + 2633x_{15} + 2634x_{16} + 2635x_{17} + 2635x_{18} + 3287x_{19} \\ +3287x_{20} + 3287x_{21} + 3289x_{22} + 3292x_{23} + 3293x_{24} + 3293x_{25} \\ = 28981,\ x_i \in \{0, 1\}. \end{aligned}$$

Similar looking DKPs

- The second one has $t = 1$, and takes $\approx 22,000$ nodes to prove infeasibility.

- The first one has $t = 2$, and takes ≈ 3.6 million nodes to prove infeasibility. (Note that $2^{25} \approx 33\,\mathrm{million}$).

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- a "not thin" direction beats a thin direction!

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- if we fix $\mathbf{p}_i^T\mathbf{x}$, the problem simplifies ($\mathbf{p}_i M_i$ disappears)
- width in direction of $\mathbf{p}_{s+1}\mathbf{x}$, after we branched on $\mathbf{p}_1\mathbf{x}, \ldots, \mathbf{p}_s\mathbf{x}$ is

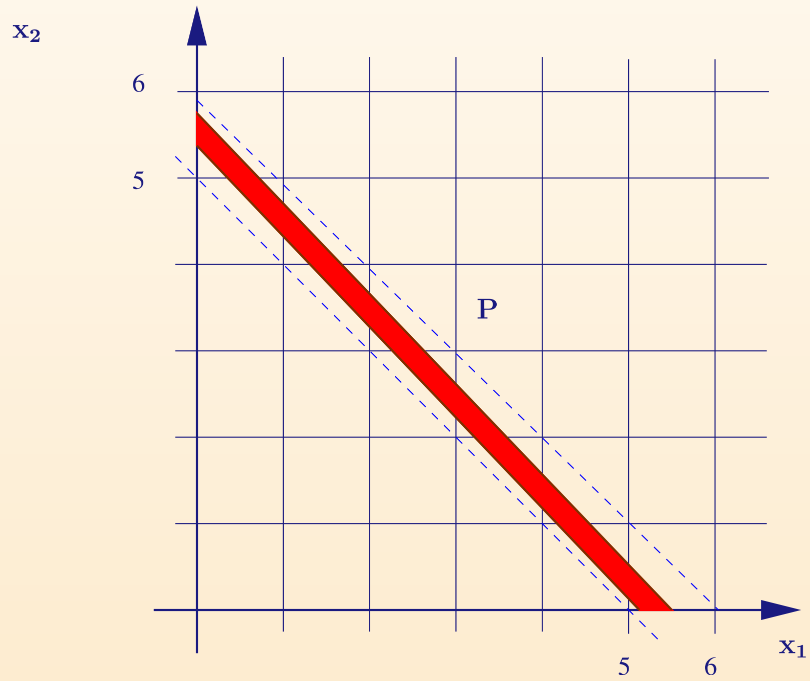
$$O\left(\frac{rhs}{M_{s+1}^2} + \frac{\delta}{M_{s+1}}\right).$$

CBR's action on DKPs

- Briefly: the "good reasons" for $\mathbf{p}_i\mathbf{x}$ are transferred to the variable y_{n-i} in the reformulation

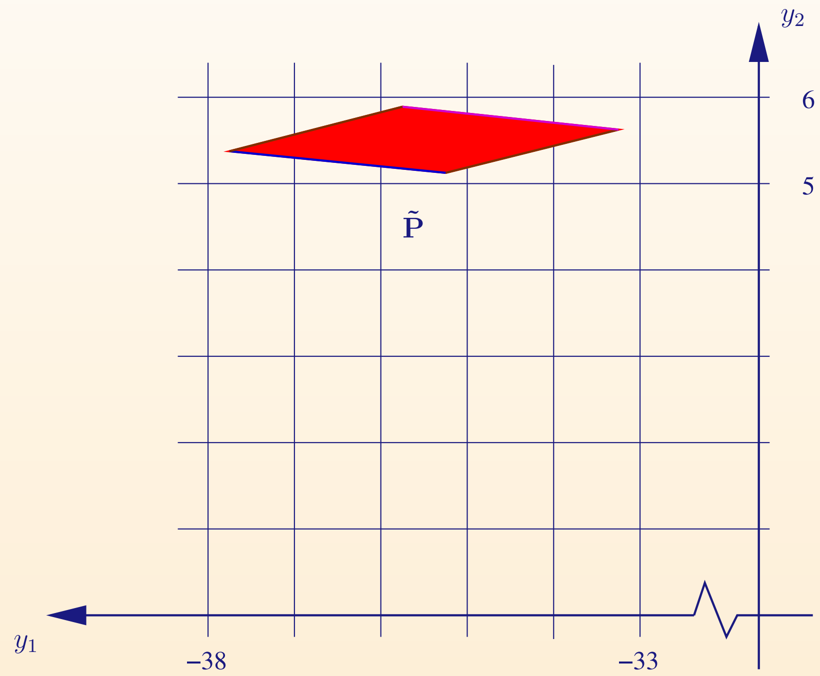
Example of CBR of a DKP

$$\begin{array}{rcl} 106 \leq & 21x_1 + 19x_2 & \leq 113 \\ x_1, x_2 \in & & \in [0,6] \cap \mathbb{Z} \end{array}$$



Hard for branching on x_is. Easy for branching on $x_1 + x_2$: $\max = 5.94$, $\min = 5.04$.

After Reformulation ...



. . . branching on y_2 proves infeasibility!

CBR's Action on DKP

- we compute U so that

$$\begin{pmatrix} \sum_{i=1}^{t} \mathbf{p}_i M_i + \mathbf{r} \\ I \end{pmatrix} U \quad \text{is reduced.}$$

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- **Theorem:** If separation between $M_1 > M_2 > \cdots > M_t$ is suitably large, then

$$\begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \vdots \\ \mathbf{p}_t \end{pmatrix} U = \begin{pmatrix} 0 & 0 \ldots & 0 & 0 & 0 & * \\ 0 & 0 \ldots & 0 & 0 & * & * \\ \vdots & & & & & \\ 0 & 0 \ldots & * & \ldots & * & * \end{pmatrix}$$

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When computing U, we do not know the decomposition!

CBR's Action on DKP

- using the correspondence $U\mathbf{y} = \mathbf{x}$, we get

$$\mathbf{p}_1\mathbf{x} = \mathbf{p}_1(U\mathbf{y}) = (\mathbf{p}_1 U)\mathbf{y} = (\mathbf{p}_1 U)_n y_n.$$

- **Corollary:**
  - Branching on y_n in reformulation $\Leftrightarrow$ branching on $\mathbf{p}_1\mathbf{x}$ in original problem
  - Afterwards: $y_{n-1} \Leftrightarrow \mathbf{p}_2\mathbf{x}$, etc.
  - Analogous result for CBR-N

Summary of CBR

- general reformulation technique for arbitrary IPs.
- has two variants: CBR-R and CBR-N, both work in practice and can be analyzed
- a fairly general class of IPs provably hard for ordinary B&B
- the provably hard problems turn into provably easy ones: the reformulation "uncovers" the hidden, dominant directions
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- The *cascade* problems: thinner $\neq$ better!
- Pataki et al. (2010) - B&B solves "almost all" instances of CBR-R of $\{\mathbf{x} \mid \boldsymbol{\ell}_1 \leq A\mathbf{x} \leq \mathbf{u}_1;\ \boldsymbol{\ell}_2 \leq \mathbf{x} \leq \mathbf{u}_2\}$ at root node if $A_{ij} \in U\{1, \ldots, M\}$ for ***sufficiently large*** M

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- $\triangle^* = 0$ (or $\triangle^* = 1$ when α odd) gives a *perfect* partition

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  - phase transition (fully characterized mathematically)
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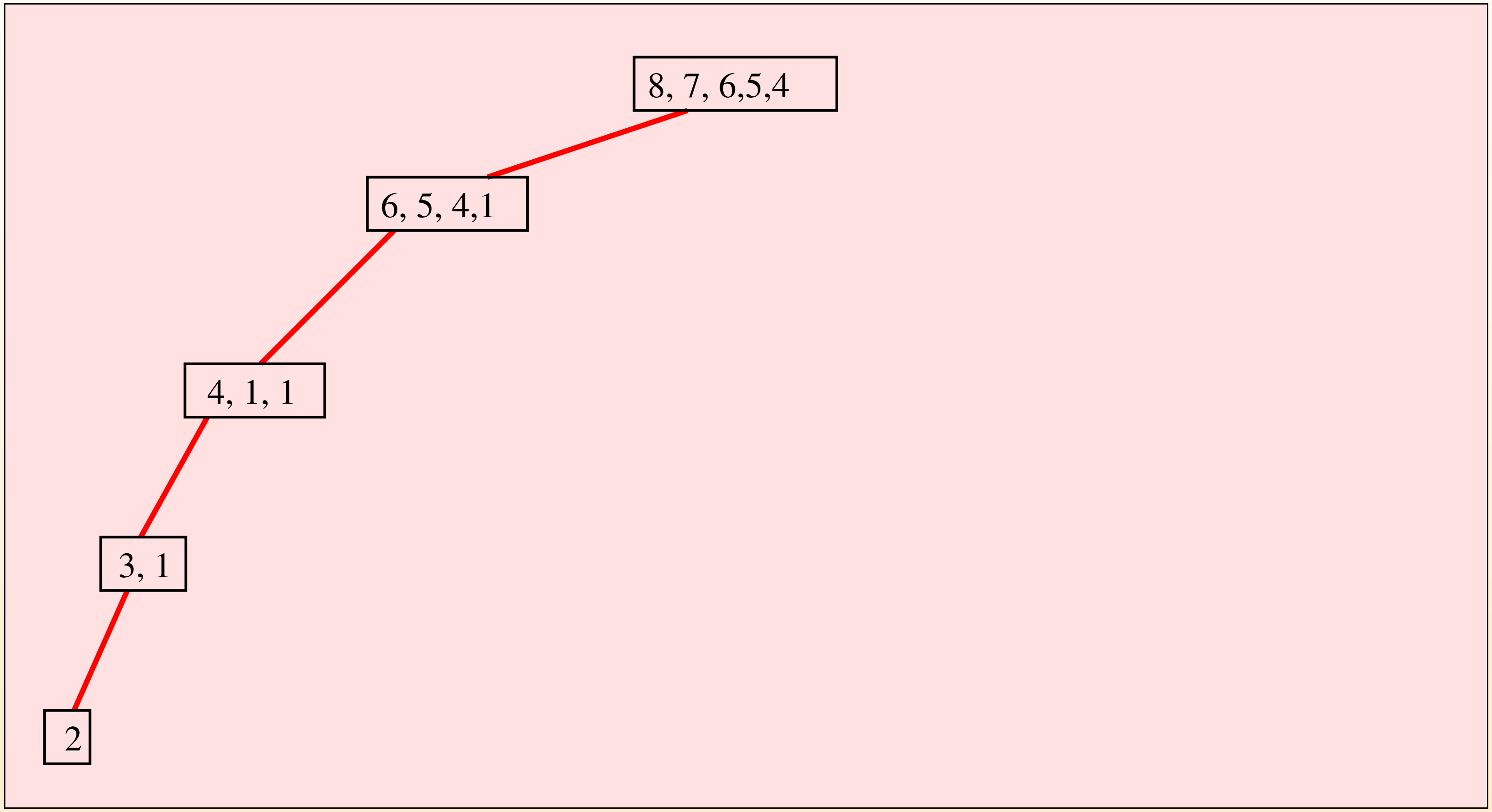

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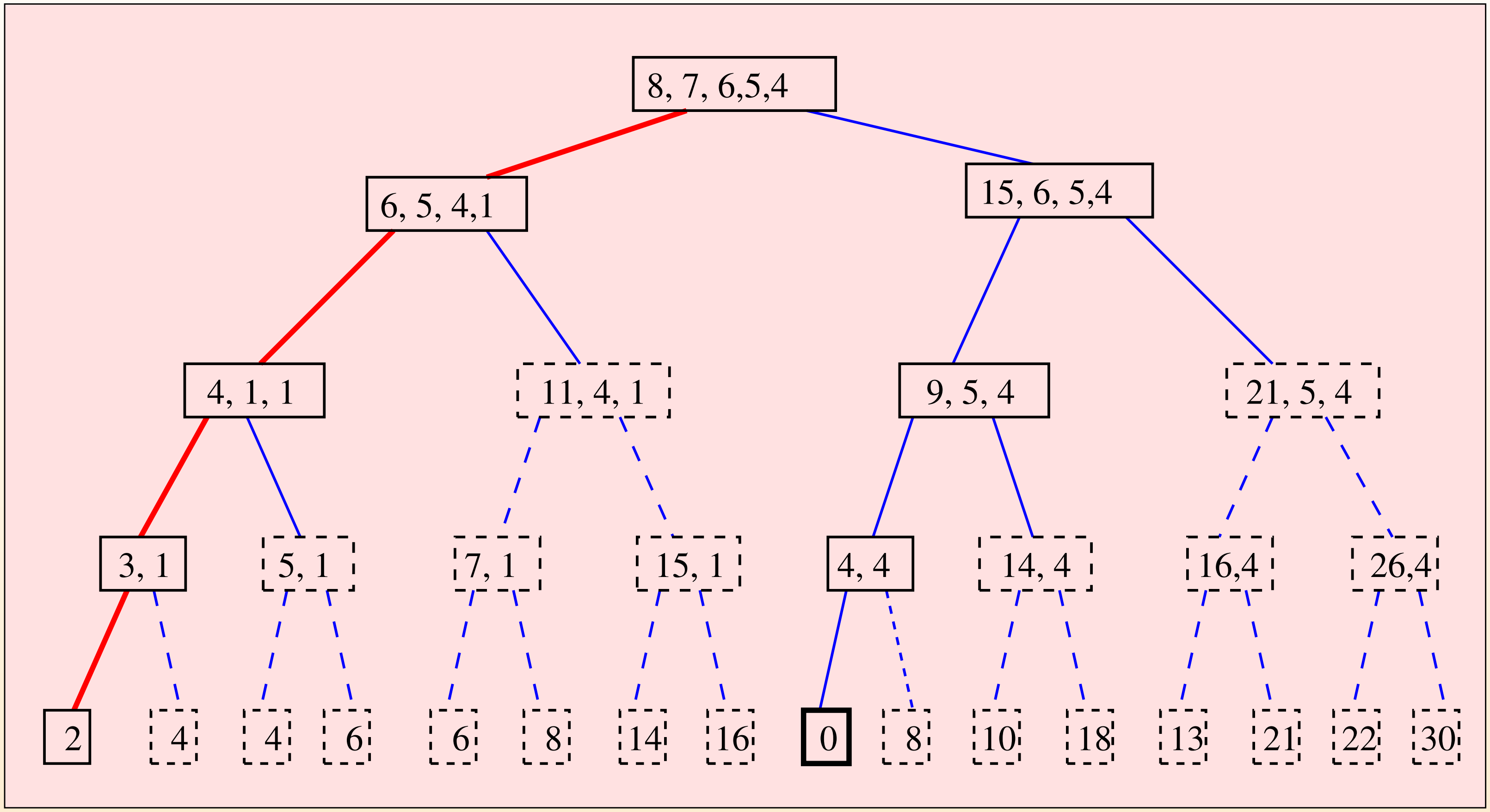

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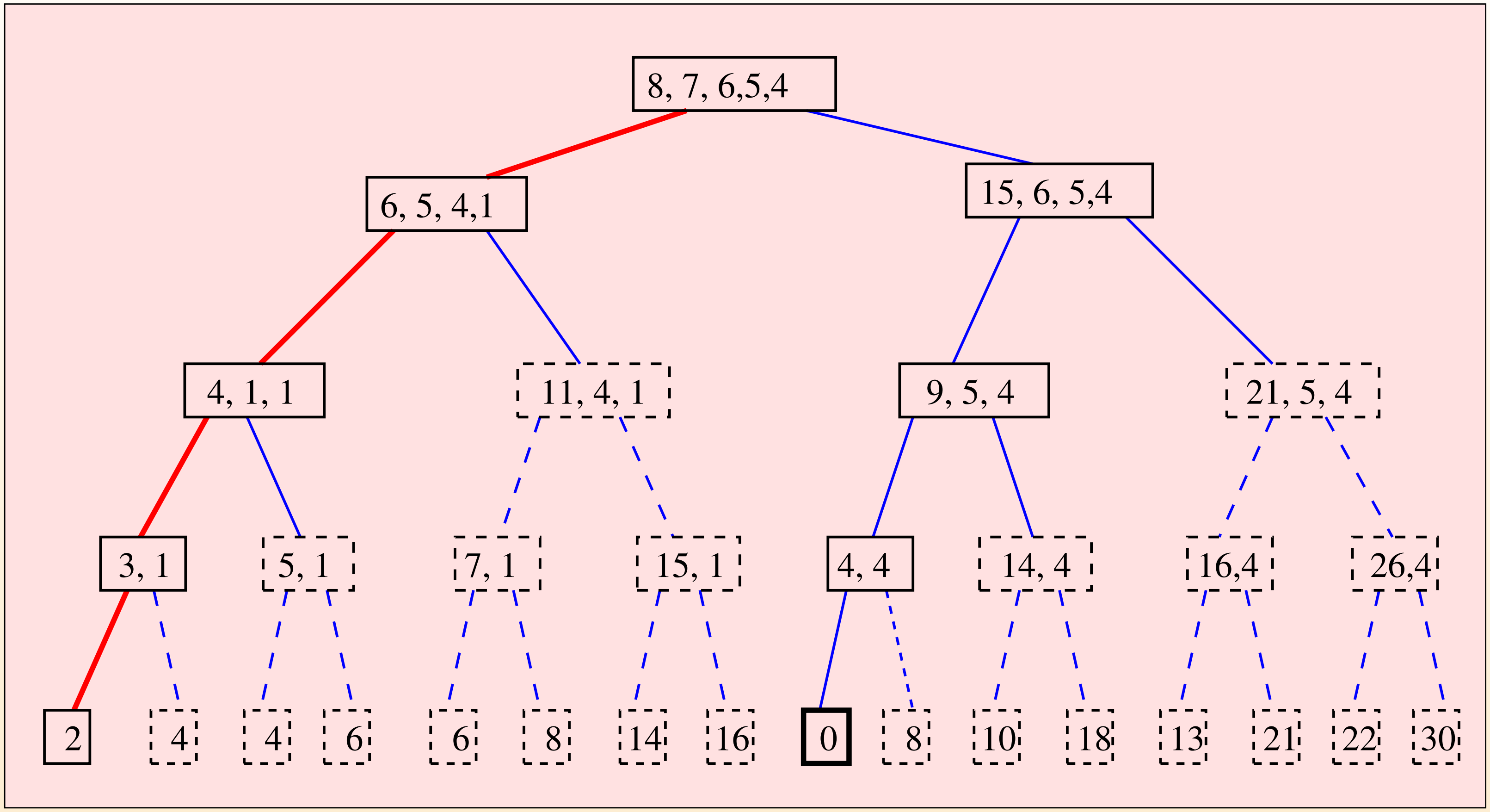

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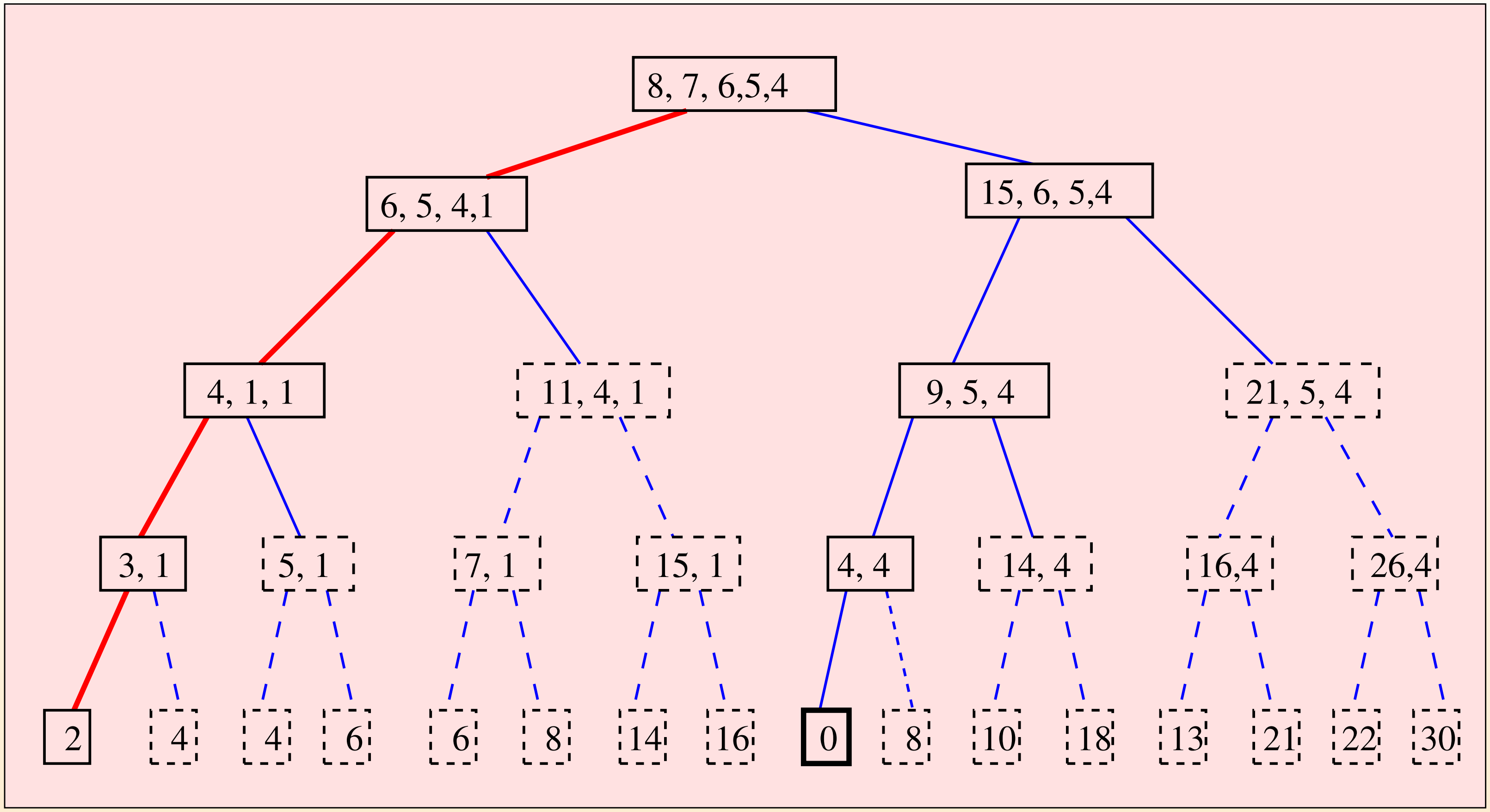

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- two-color associated tree to recover partition

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- typical numbers are *huge*; for $n = 30$, look at a_j's with 11 digits!

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- Lagarias & Odlyzko (85), Coster et al. (92): for subset sums

BR Algo Tests:

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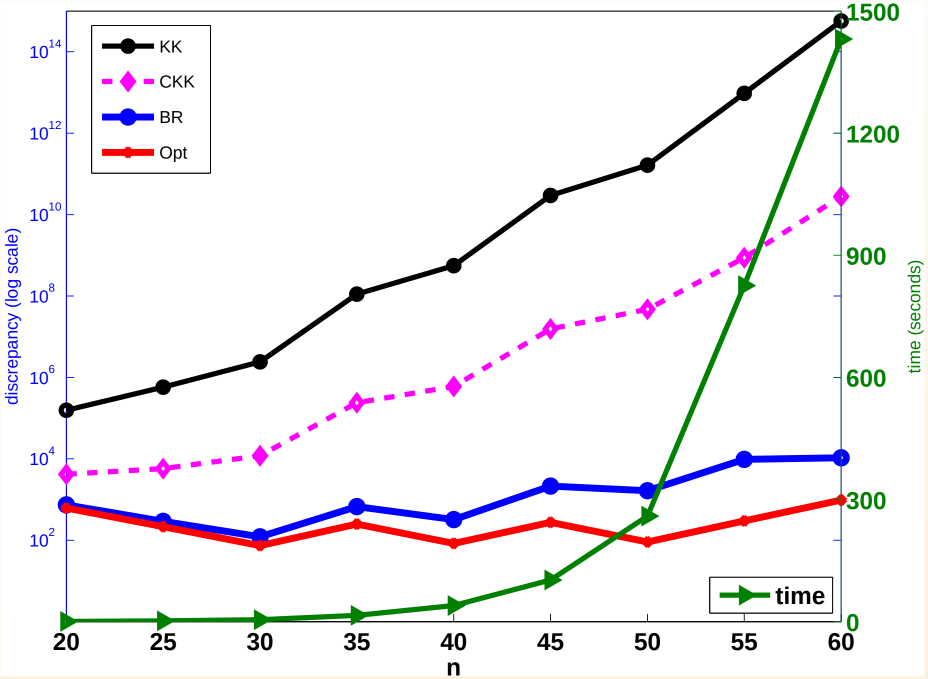

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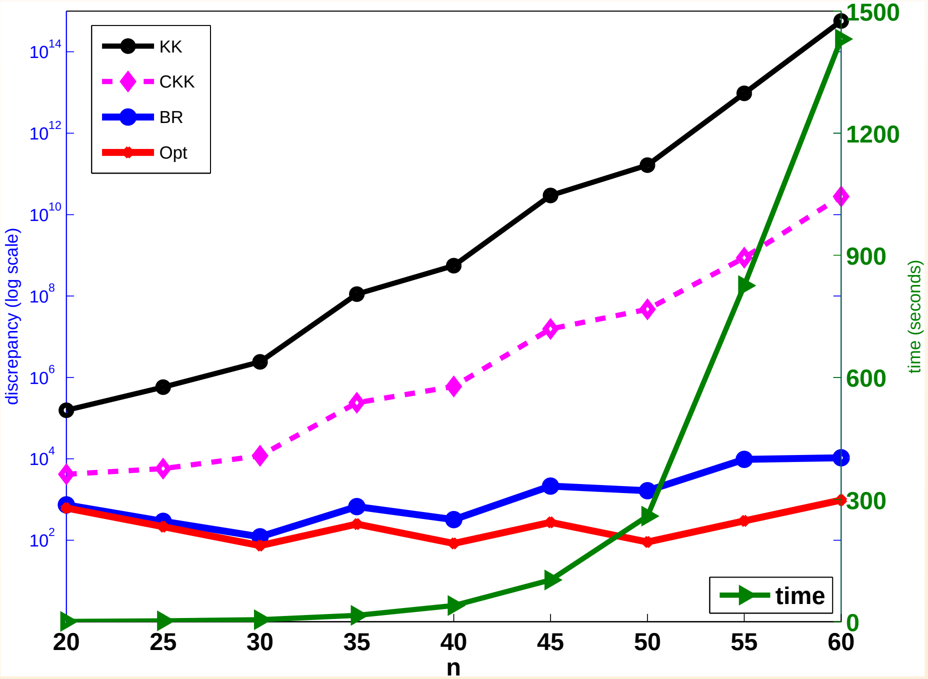

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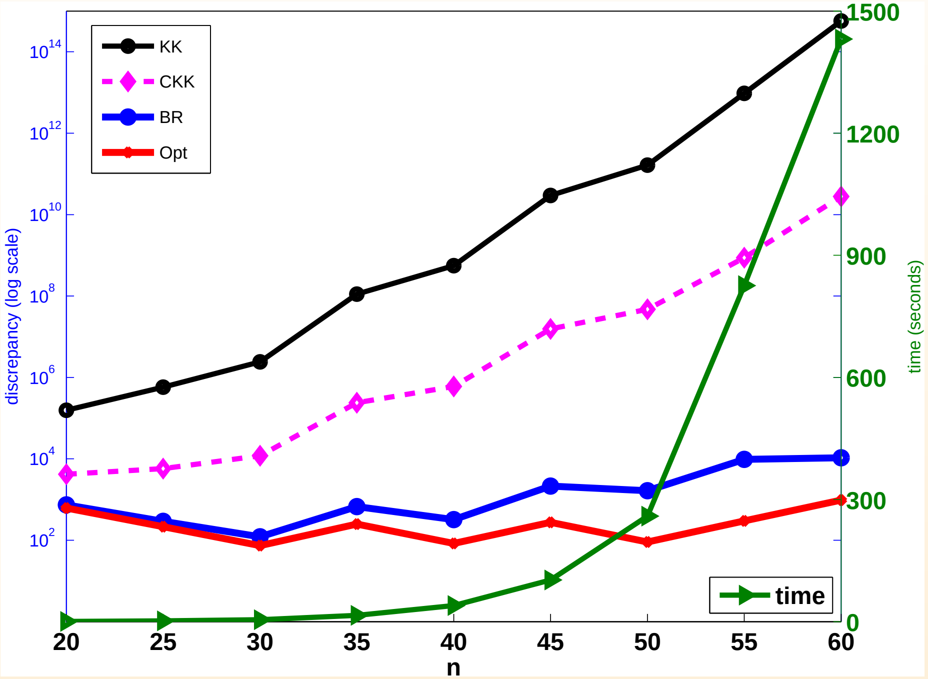
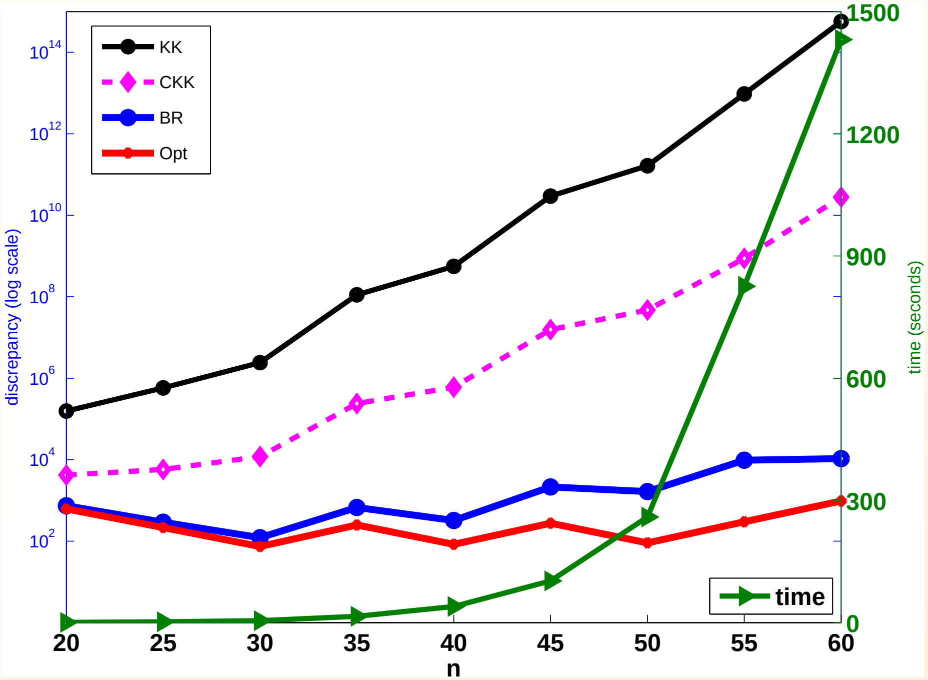
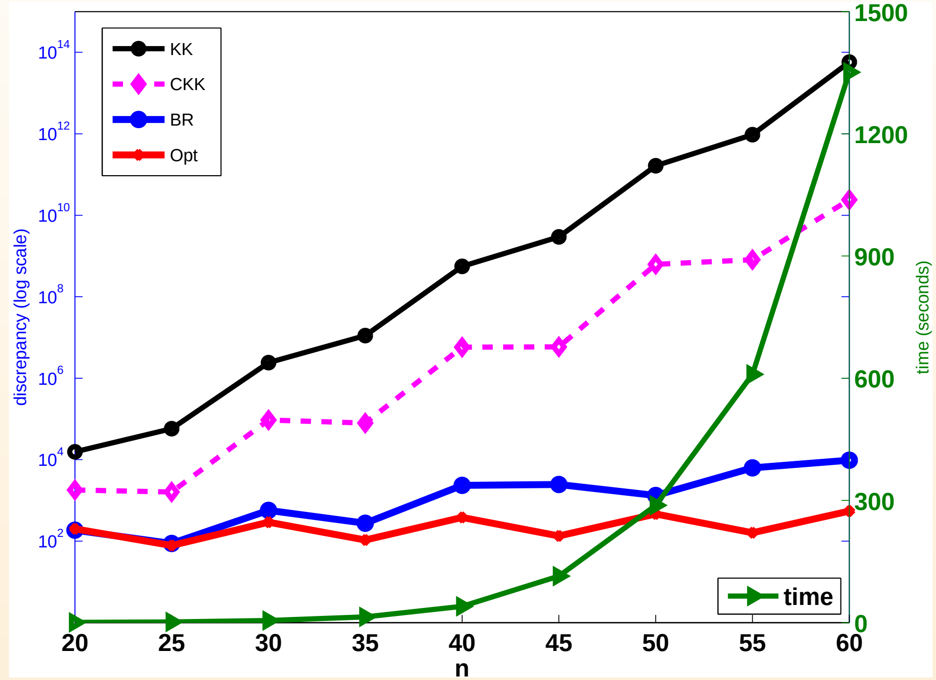


- block Korkine-Zolotarev (BKZ) reduction
- opt: $\triangle^* = \sqrt{n}\, 2^{-n} R$ is plotted
- ckk: estimated $\triangle_{CKK}$ for same running time as BR

BR Algo Tests: for NPP and BALNPP





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  MIP for NPP:

$$\begin{array}{llcll}
\min & 2w & & & \\
\text{s.t.} & w & \geq & \sum a_j x_j - \beta & \\
 & w & \geq & -\sum a_j x_j + \beta & \\
 & x_j & \in & \{0,1\} & j = 1, \ldots, n.
\end{array}$$

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- solve the CBR-R reformulation using standard solver:

$$\min\{\, w \mid \tilde{A}\mathbf{y} + Bw \le \tilde{\mathbf{b}},\ \mathbf{y} \in \mathbb{Z}^n \,\}$$

CBR-R Tests:

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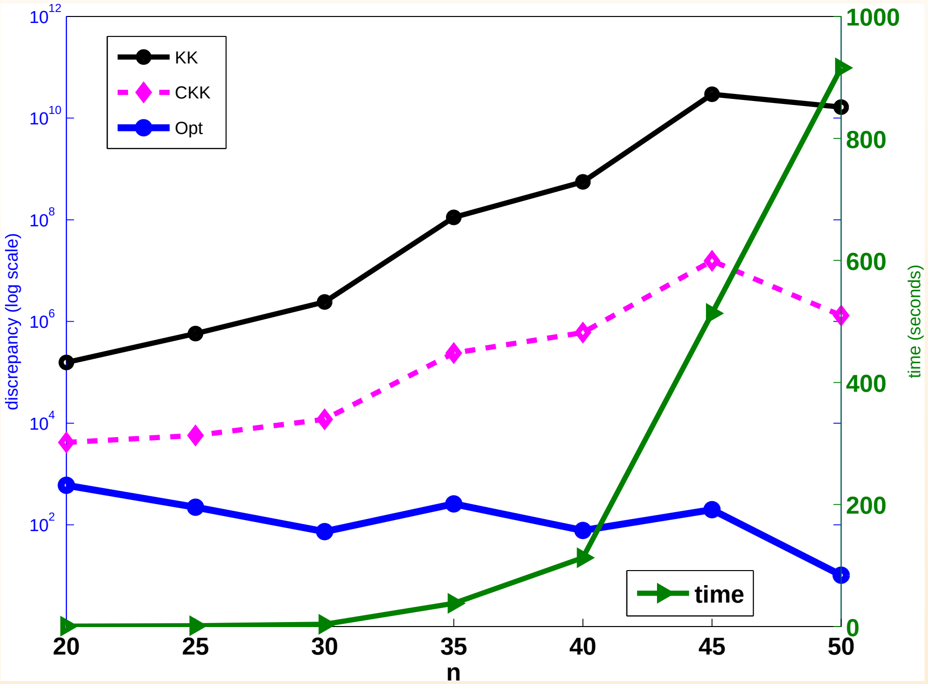

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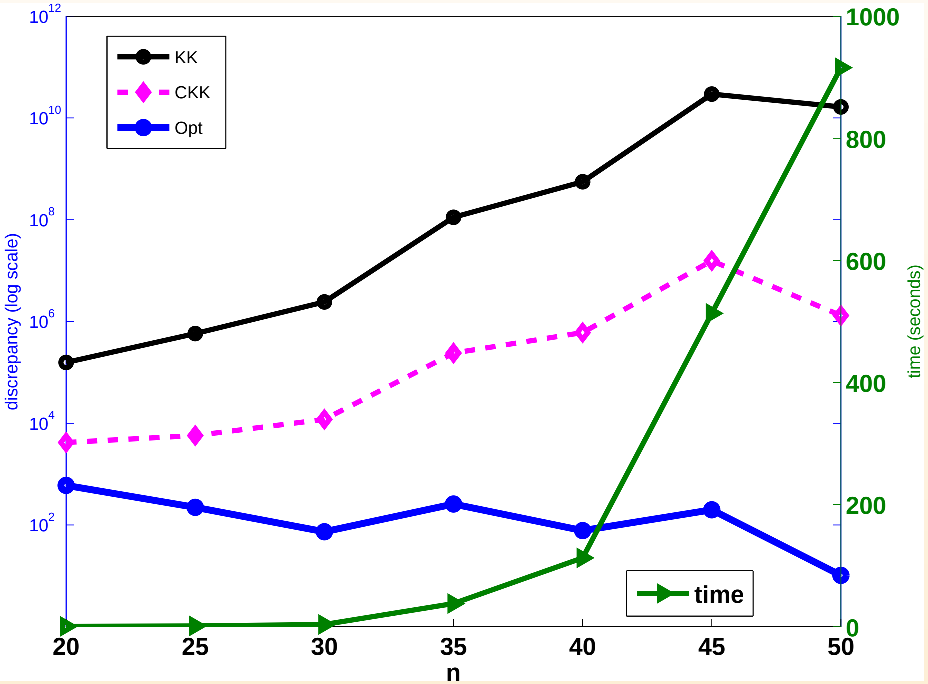

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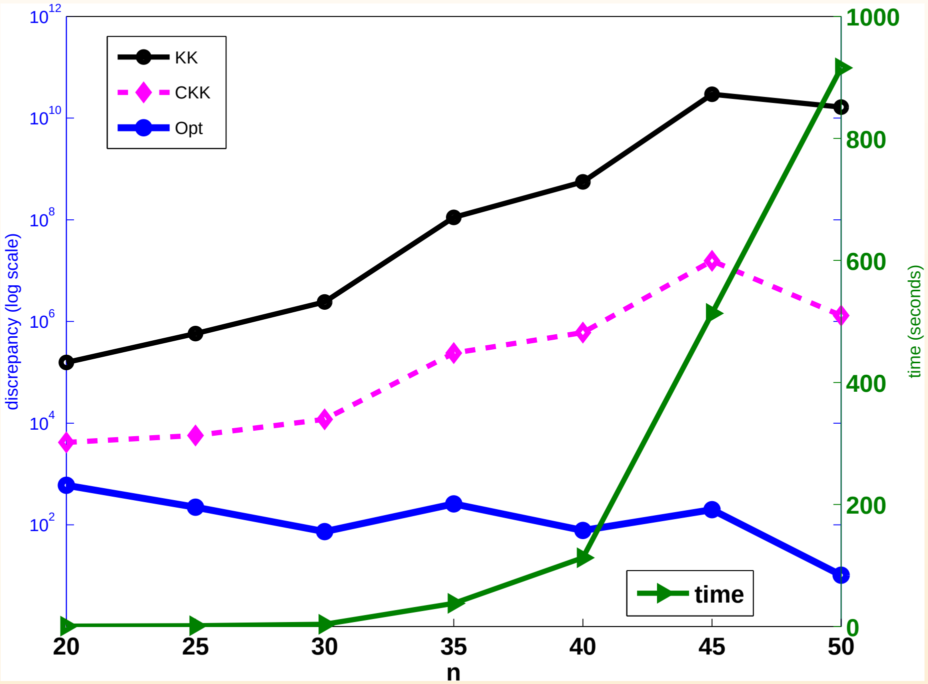

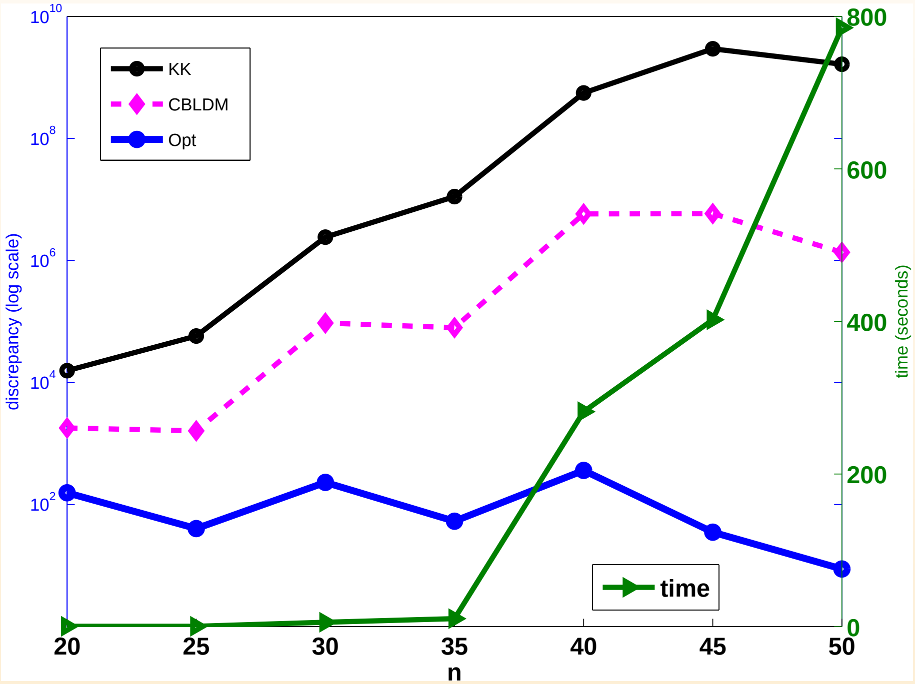

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- running times increase with R
- for $n \geq 100$, KK may still be the best (current) option

Outline

- Number Partitioning Problem (NPP)
- Karmarkar-Karp differencing (KK)
- NPP and the Closest Vector Problem (CVP)
- A Basis Reduction Heuristic for NPP
- Mixed Integer Program (MIP) for NPP