Optimal Homologous Cycles, Total Unimodularity, and Linear Programming

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joint work with

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and

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U. Illinois

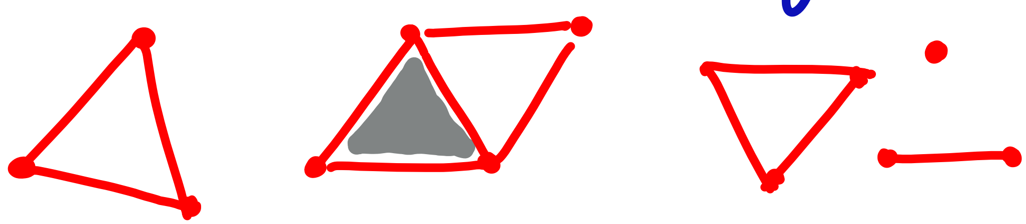
(to appear in STOC '10)

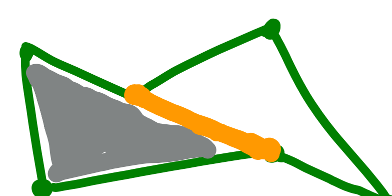
SIMPLICIAL COMPLEX

K: a collection of simplices in $\mathbb{R}^d$ such that

(1) every face of a simplex in K is in K;

(2) intersection of two simplices of K is a face of each of them.

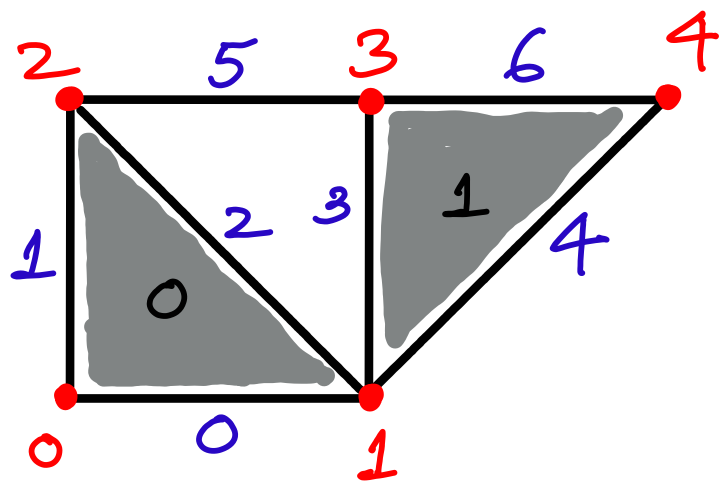




not a simplicial complex

MOTIVATING EXAMPLE

K:



hole in the middle

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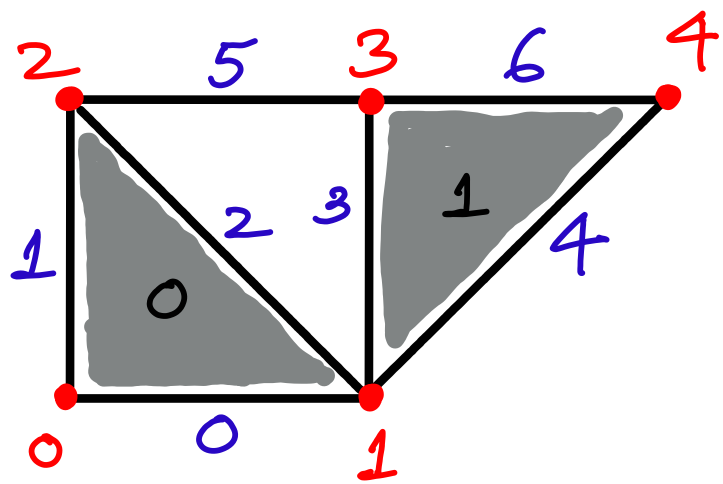

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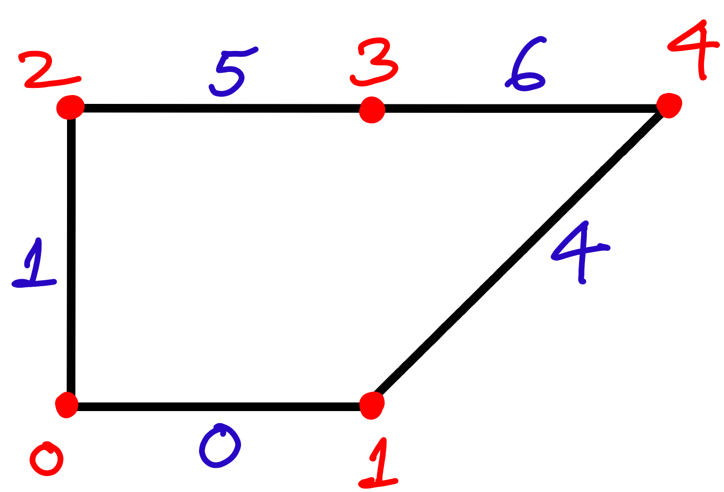
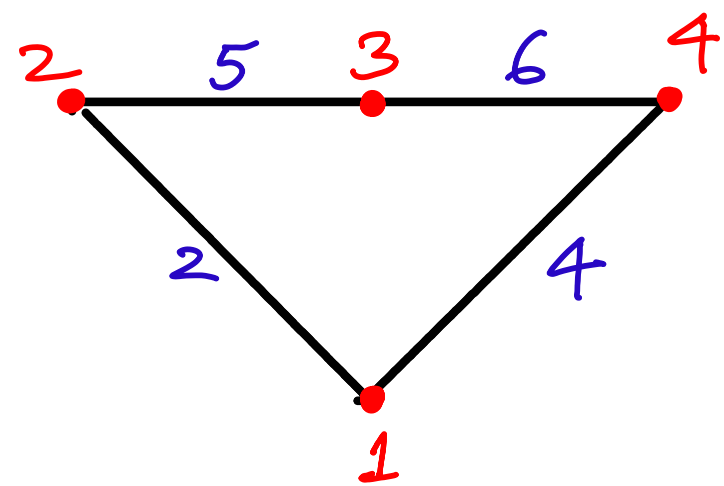
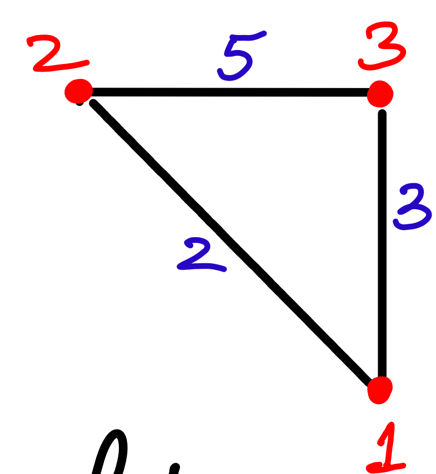
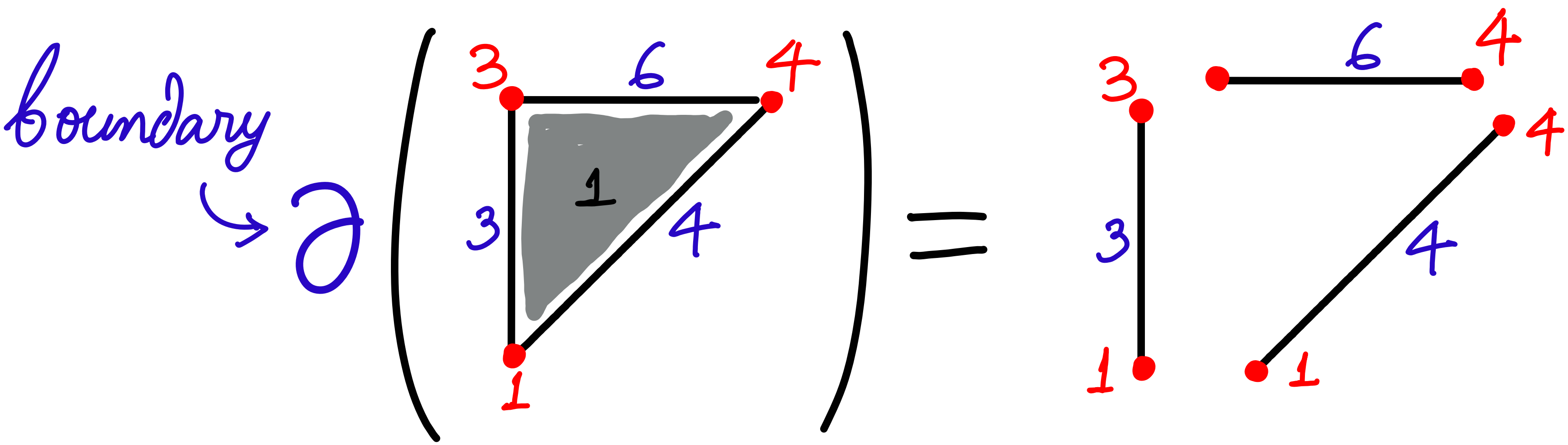






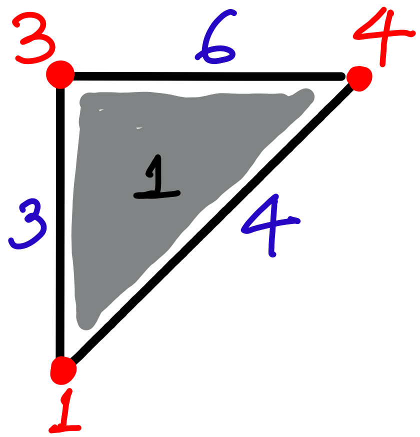
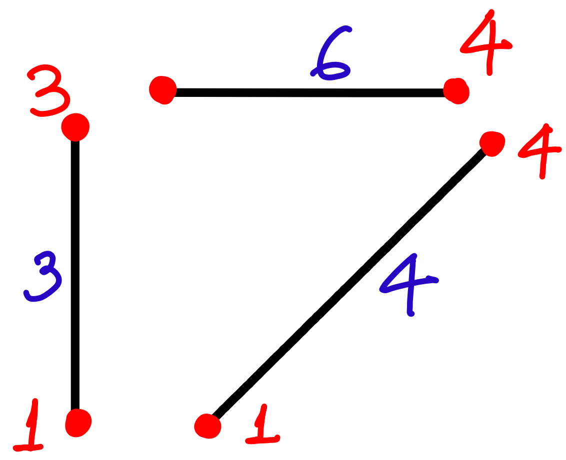
3 cycles representing the same hole

MOTIVATING EXAMPLE

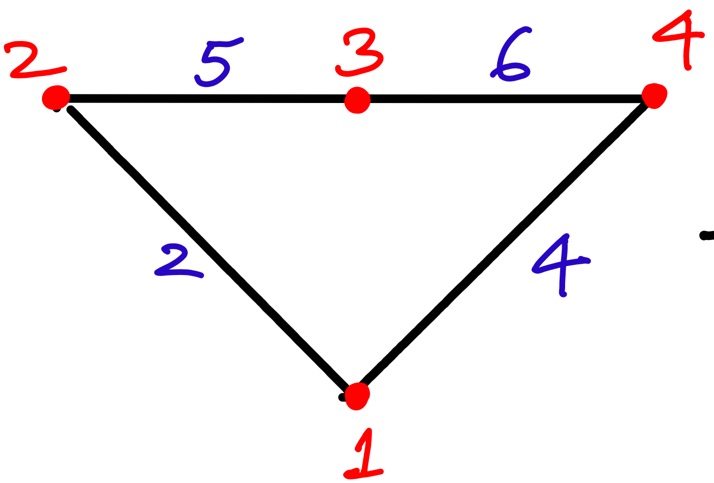
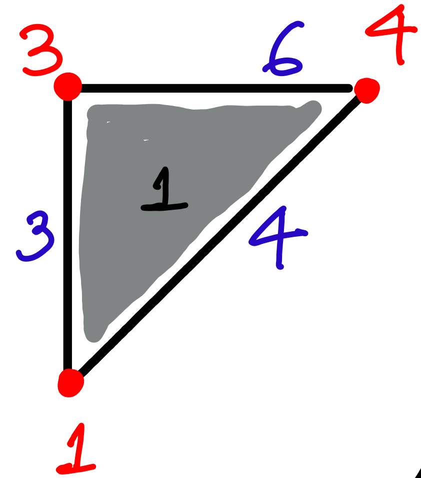
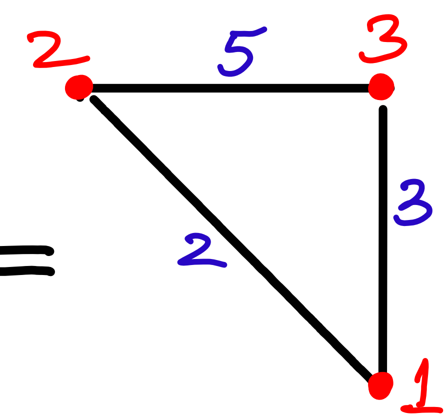


MOTIVATING EXAMPLE

boundary

∂ () = 

add edges in $\mathbb{Z}_2$ $(1+1=0)$

 $+\ \partial$ () = 

3

OUR RESULT

- Problem is NP-hard with addition over $\mathbb{Z}_2$

OUR RESULT

- ✗ Problem is NP-hard with addition over $\mathbb{Z}_2$

- ✓ With addition over $\mathbb{Z}$, can solve the problem in polynomial time for a large majority of K using linear programming

ABSTRACT SIMPLICIAL COMPLEX

A collection of finite, non-empty sets $\mathcal{S}$, such that if $A \in \mathcal{S}$, then $B \in \mathcal{S}$, $\forall B \subseteq A$

$$\mathcal{S} = \{\{0\},\{1\},\{2\},\{3\},\{4\},\{0,1\},\{0,2\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{3,4\},\{0,1,2\},\{1,3,4\}\}$$

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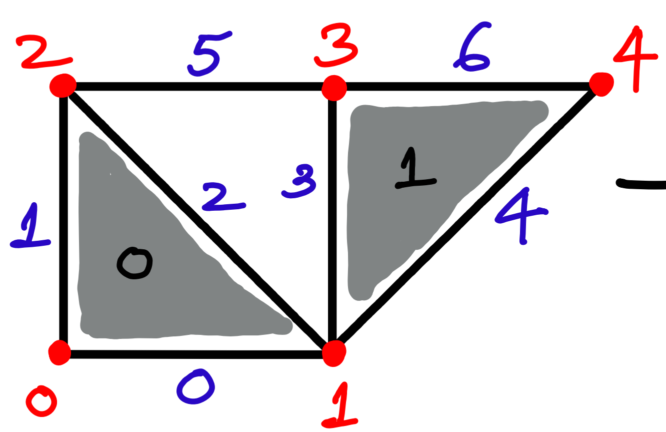
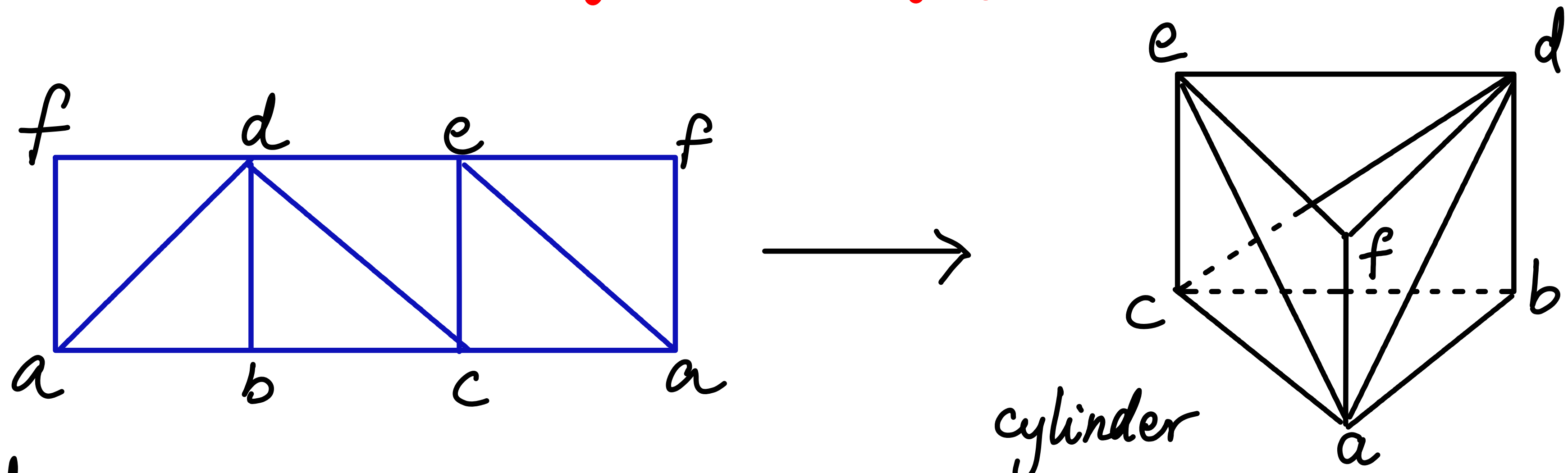
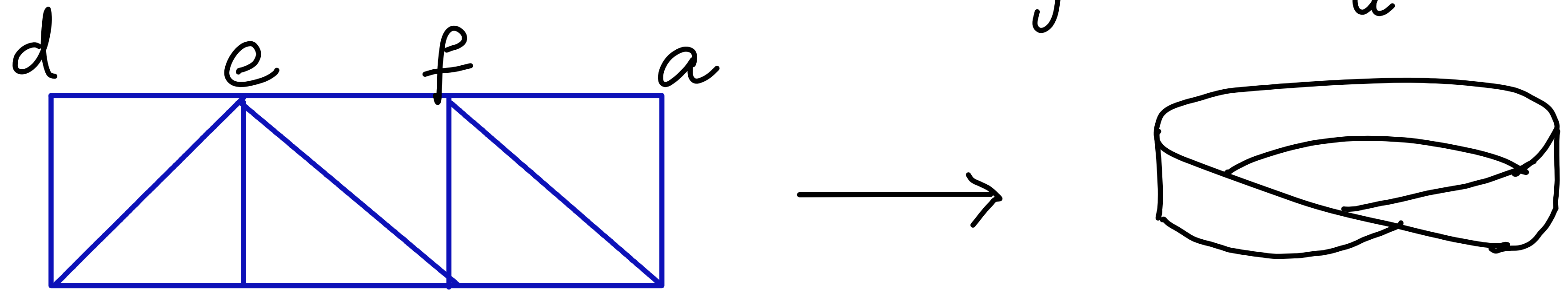


$\longrightarrow$ a geometric realization

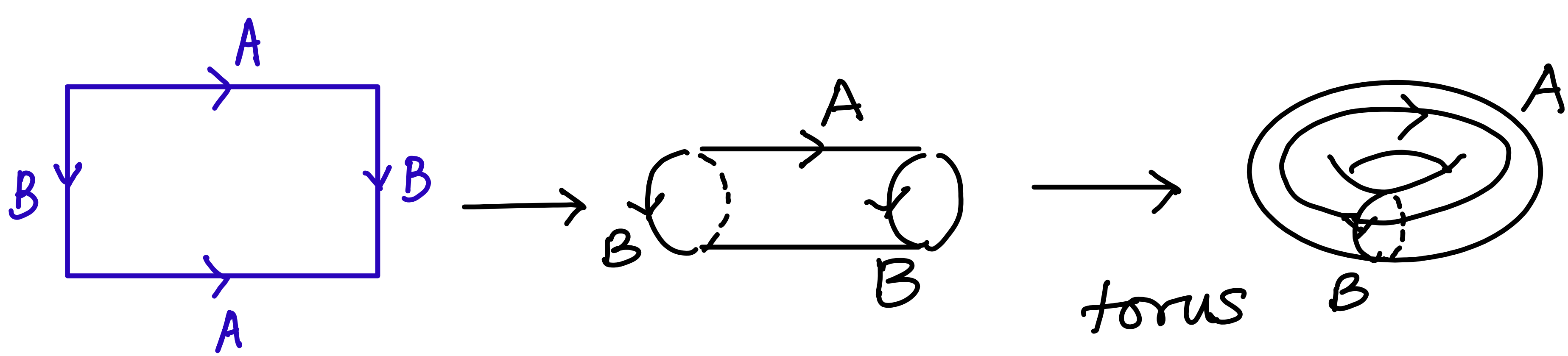
EXAMPLES



cylinder



Möbius band



torus

CHAINS

1-chain:

collection of edges

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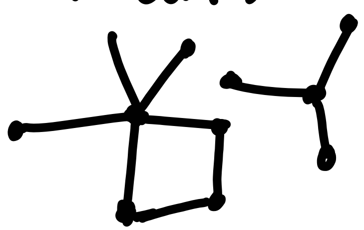
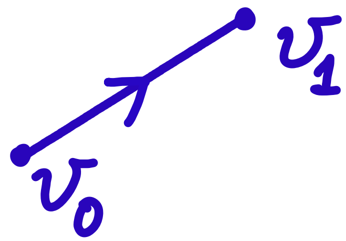
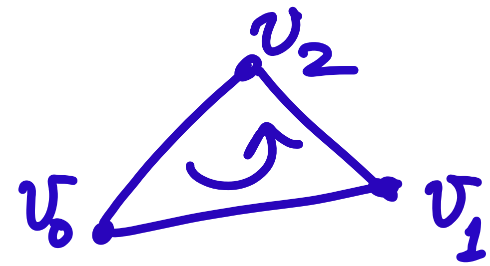


collection of edges

Orientation of a simplex:



$[v_0 v_1]$ or $[v_1 v_0]$



$[v_0 v_1 v_2]$ or $[v_0 v_2 v_1]$

CHAINS

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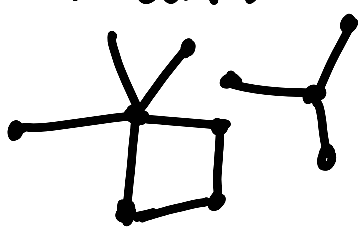

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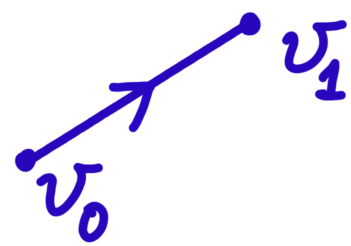

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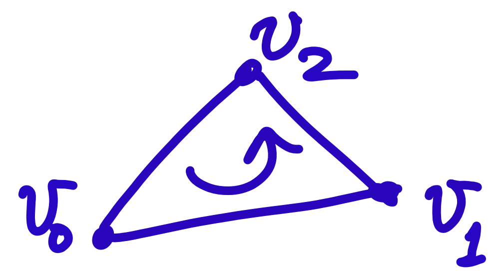

$[v_0v_1v_2]$ or $[v_0v_2v_1]$

p-Chain: A function c from a set of oriented p-simplices of K to $\mathbb{Z}$ s.t.:

(1) $c(\sigma) = -c(\sigma')$ if σ, σ' are opposite orientations of same simplex;

(2) $c(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

CHAIN GROUPS

Add p-chains by adding their values over $\mathbb{Z}$ $\Rightarrow$
$C_p(K)$: group of (oriented) p-chains.

Elementary chain of $\sigma \in K$:

$c(\sigma) = 1$,
$c(\sigma') = -1$, if σ': opposite orientation of σ
$c(\tau) = 0 \quad \forall \tau \neq \sigma, \sigma'$.

Result: $C_p(K)$ is free abelian; the elementary chains form a basis for $C_p(K)$.

BOUNDARY OPERATOR

The homomorphism $\partial_p : C_p(K) \to C_{p-1}(K)$.

$\sigma = [v_0, \dots, v_p]$: oriented simplex, $p > 0$.

$$\partial_p \sigma = \partial_p [v_0, \dots, v_p] = \sum_{i=0}^{p} (-1)^i [v_0, \dots, \hat{v}_i, \dots v_p]$$

delete v_i

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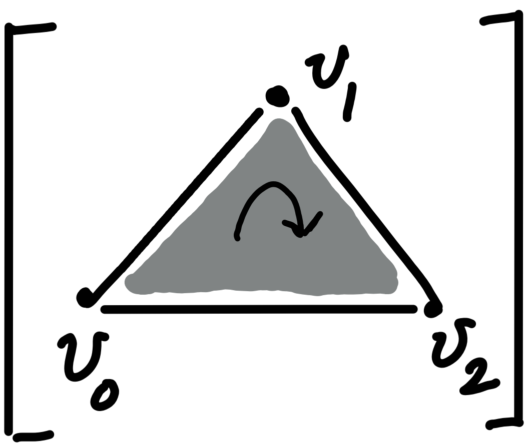
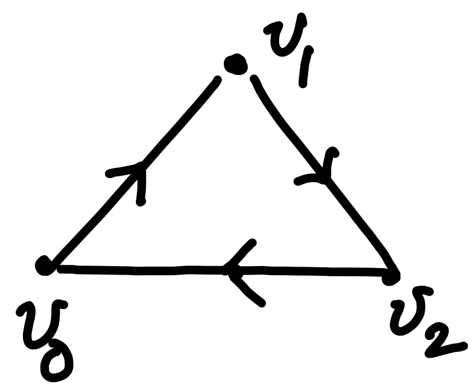
delete v_i

e.g., $\partial_2 [v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

∂_2 

$=$ 

HOMOLOGY GROUPS

Lemma: $\partial_{p-1} \circ \partial_p = 0$ boundary of boundary is empty

$\ker \partial_p = Z_p(K)$ group of p-cycles

$\operatorname{im} \partial_{p+1} = B_p(K)$ group of p-boundaries

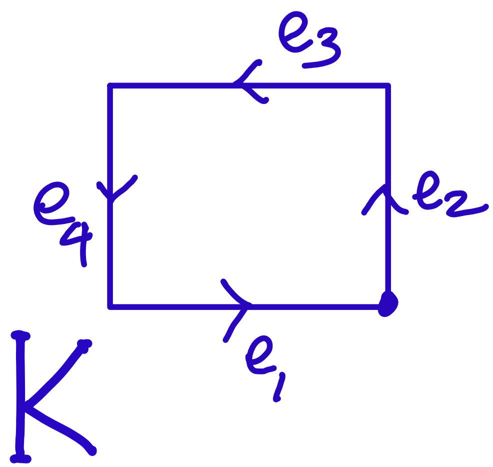
$$B_p(K) \subset Z_p(K) \subset C_p(K)$$

$$H_p(K) = Z_p(K)/B_p(K)$$

group of p-cycles that are NOT p-boundaries

p^{th} homology group of K.

EXAMPLE



$C_1(K)$: free abelian of rank 4

general 1-chain: $c = \sum_{i=1}^{4} n_i e_i$

c is a cycle $\Longleftrightarrow n_1 = n_2 = n_3 = n_4$

$\Rightarrow Z_1(K)$ is infinite cyclic, generated by $e_1 + e_2 + e_3 + e_4$

No 2-simplices in $K \Rightarrow B_1(K)$ is trivial.

$\Rightarrow H_1(K) = Z_1(K)/B_1(K) \cong \mathbb{Z}$.

RANK OF $H_p(K) = \beta_p(K)$

Betti numbers of K:

Intuitively, $\beta_0 = \#$ connected components

$\beta_1 = \#$ holes

$\beta_2 = \#$ tunnels/voids

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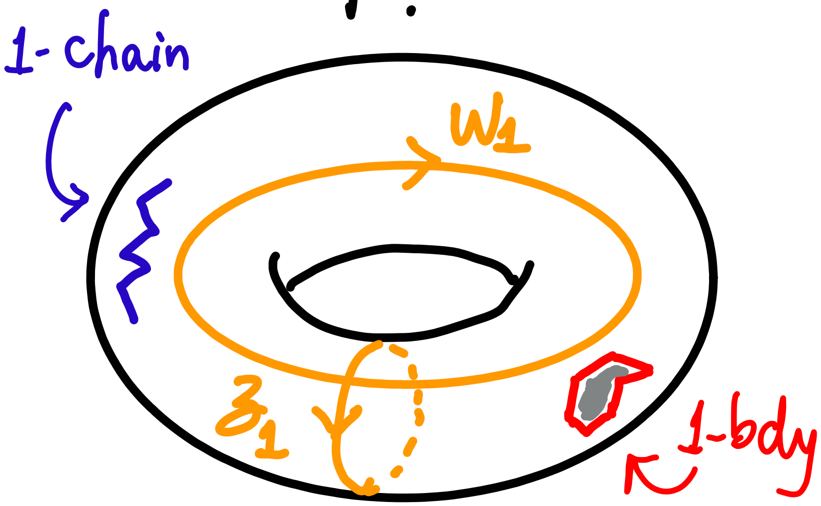

$\beta_0 = 1, \beta_1 = 2, \beta_2 = 1.$

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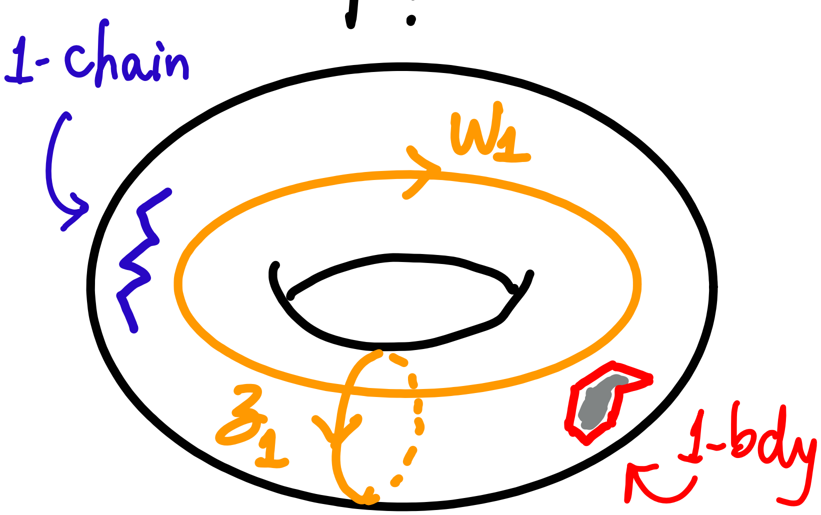

$\beta_0 = 1, \beta_1 = 2, \beta_2 = 1.$

Could study $H_p(K, G)$ for $G = \mathbb{Z}, \mathbb{Z}_2, \mathbb{Q}$, etc.

$\mathbb{Z}_2$: widely used for computation. field, simple, intuitive

BOUNDARY MATRIX $[\partial_p]$

$\partial_p : C_p(K) \to C_{p-1}(K)$

If $\{\sigma_i\}_{i=0}^{m-1}$ and $\{\tau_j\}_{j=0}^{n-1}$ are elementary chain bases for $C_{p-1}(K)$ & $C_p(K)$, then $[\partial_p]$ is an $m \times n$ matrix, $[\partial_p]_{ij} \in \{-1, 0, 1\}$.

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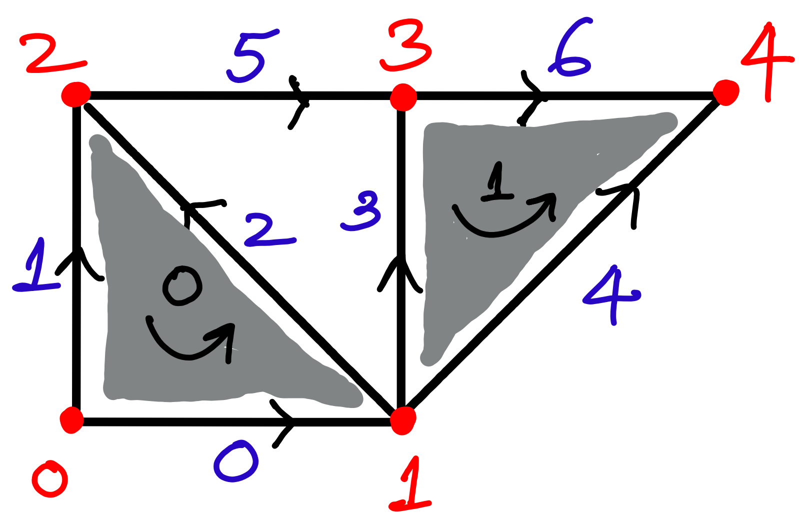
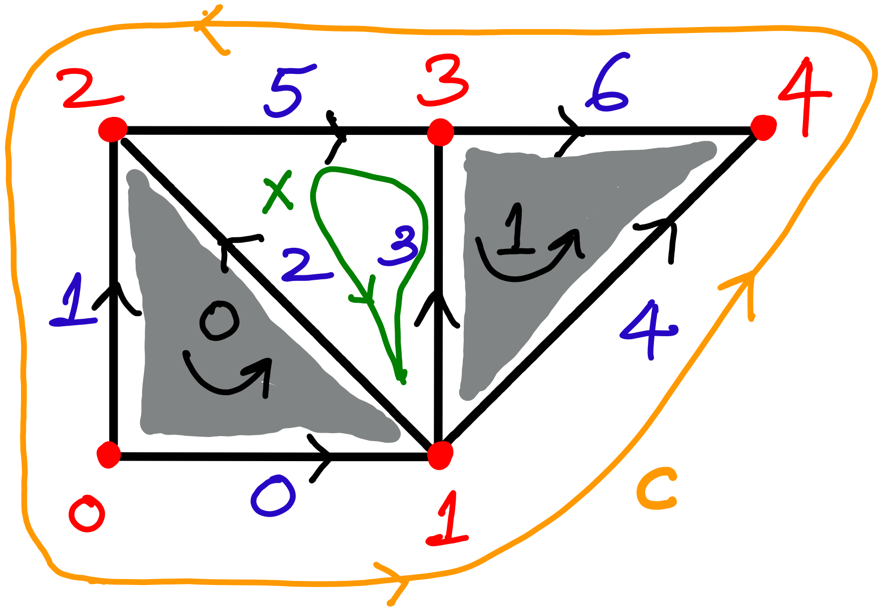


$$[\partial_2] = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \end{array}$$

SHORT HOMOLOGOUS CYCLES



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represents hole in middle, but has 5 edges.

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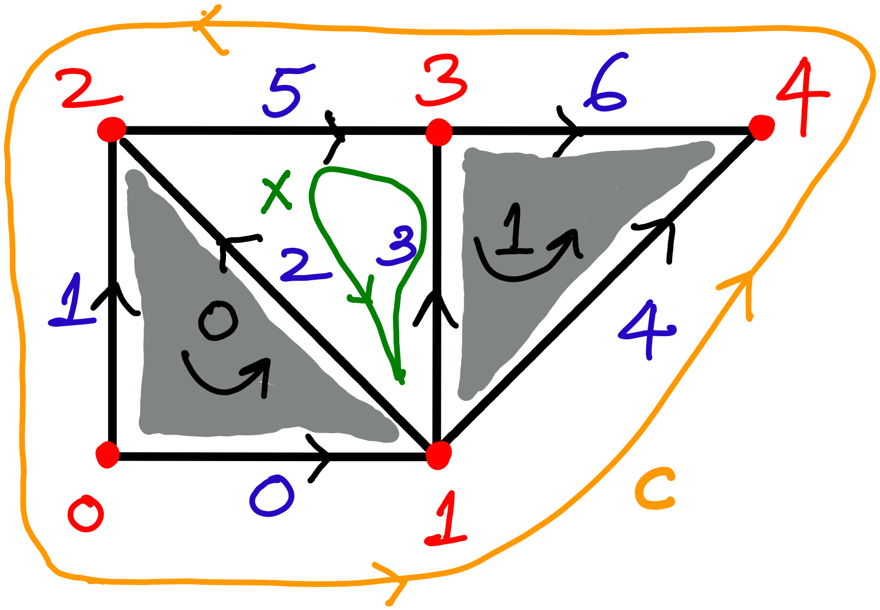


$$c = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

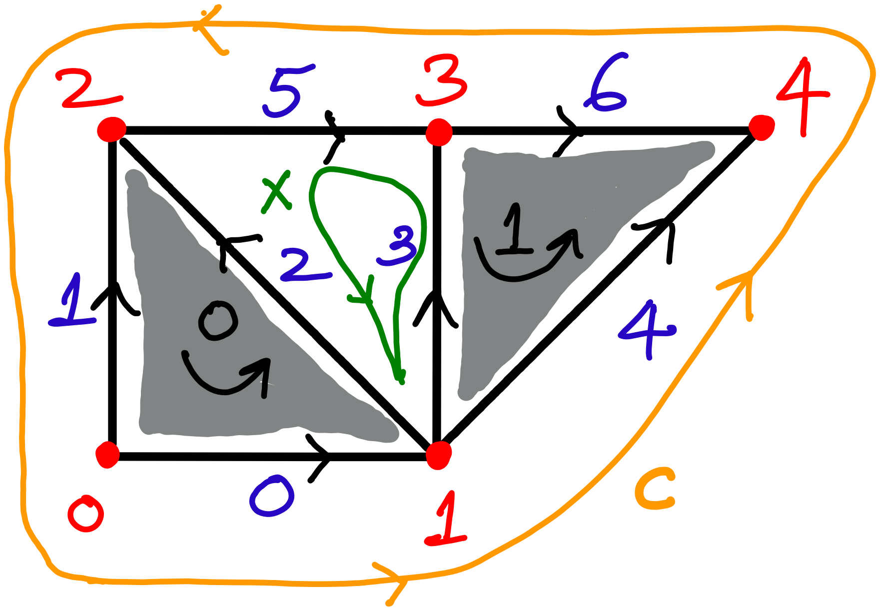
represents hole in middle, but has 5 edges.

$$x = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

is homologous to c, but is shorter (has only 3 edges)

x is "tightest" cycle around the hole

Short Homologous Cycles



$$c = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} \qquad x = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

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$$x = c + [\partial_2]\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$x \sim c$ (x is homologous to c)

Can study weighted (in $\mathbb{R}$) chains/cycles (instead of ± 1 weights)

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OPTIMAL HOMOLOGOUS CYCLE PROBLEM

OHCP: Given a p-cycle c in K, find a cycle c^* with smallest value of $\|Wc^*\|_1$ among all cycles homologous to c.

$W = \text{diag}([w_1, \ldots, w_m])$, where $w_i \in \mathbb{R}_{\geq 0}$ is the weight of p-simplex $\sigma_i \in K$.

Optimal Homologous Cycle Problem

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With homology defined over $\mathbb{Z}_2$, OHCP is NP-hard (Chen & Freedman, 2010)

Optimal Homologous Chain Problem

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OHCP AS AN INTEGER PROGRAM

$$\min_{x,y} \|Wx\|_1 \quad \text{such that}$$

$$x = c + [\partial_{p+1}]\,y \;, \quad x \in \mathbb{Z}^m, \; y \in \mathbb{Z}^n$$

OHCP as an Integer Program

$$\min_{x,y} \|Wx\|_1 \quad \text{such that} \qquad = \sum_i |w_i||x_i|$$

piecewise linear

$$x = c + [\partial_{p+1}]y \;, \quad x \in \mathbb{Z}^m, \; y \in \mathbb{Z}^n$$

$$\min \sum_i |w_i|(x_i^+ + x_i^-) \qquad \text{(IP)}$$

$$\text{s.t.} \quad x^+ - x^- = c + [\partial_{p+1}]y$$

$$x^+, x^- \geq 0, \quad x^+, x^- \in \mathbb{Z}^m, \; y \in \mathbb{Z}^n$$

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ignore to get LP relaxation

IP and Total Unimodularity

$$\left.\begin{aligned} &\min\{c^T x \mid Ax = b,\ x \geq 0,\ x \in \mathbb{Z}^n\} \quad (IP) \\ &\min\{c^T x \mid Ax = b,\ x \geq 0\} \quad (LP) \end{aligned}\right\} \begin{aligned} &A \in \mathbb{Z}^{m \times n} \\ &b \in \mathbb{Z}^m \end{aligned}$$

IP and Total Unimodularity

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<u>Result:</u> (IP) can always be solved in polynomial time by solving (LP) iff A is totally unimodular.

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(Ye (1991) – $O(n^3 L)$)

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e.g., node-arc incidence matrix of a graph

OHCP AND TU OF $[\partial_{p+1}]$

$$\begin{aligned} \min \quad & \sum_i |w_i|(x_i^+ + x_i^-) \\ \text{s.t.} \quad & x^+ - x^- = c + [\partial_{p+1}]y \qquad \text{(LP)} \\ & x^+, x^- \geq 0 \end{aligned}$$

The constraint matrix of above LP is TU iff $[\partial_{p+1}]$ is TU.

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The constraint matrix of above LP is TU iff $[\partial_{p+1}]$ is TU.

$\Rightarrow$ OHCP (with homology defined over $\mathbb{Z}$) is solvable in polynomial time iff $[\partial_{p+1}]$ is TU.

LP for OHCP in $\mathbb{Z}_2$?

OHCP in $\mathbb{Z}_2$ as an IP: With $c \in \{0,1\}^m$

$$\begin{aligned}
\min \quad & \|Wx\|_1 \\
\text{s.t.} \quad & x = c + [\partial_{p+1}]y + 2u \\
& x \in \{0,1\}^m \\
& y \in \mathbb{Z}^n,\ u \in \mathbb{Z}^m
\end{aligned}$$

destroys TU.

Constraint matrix NOT TU even if $[\partial_{p+1}]$ is.

VARIANTS OF OHCP LP

Minimizing number of simplices:

$$\min_{x,y} \|x\|_1 \text{ s.t. } x = c + [\partial_{p+1}]y,\ x \in \{-1,0,1\}^m,\ y \in \mathbb{Z}^n.$$

LP:

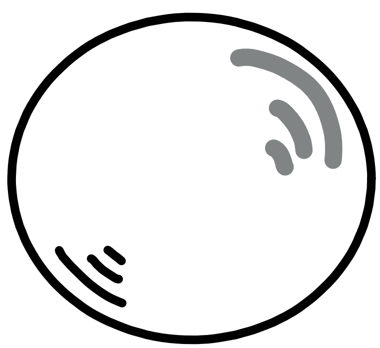
$$\begin{aligned} \min \quad & \sum_{i=1}^{m} x_i^+ + x_i^- \\ \text{s.t.} \quad & x_i^+ - x_i^- = c + [\partial_{p+1}]y \\ & x^+, x^- \leq 1 \\ & x^+, x^- \geq 0 \end{aligned}$$

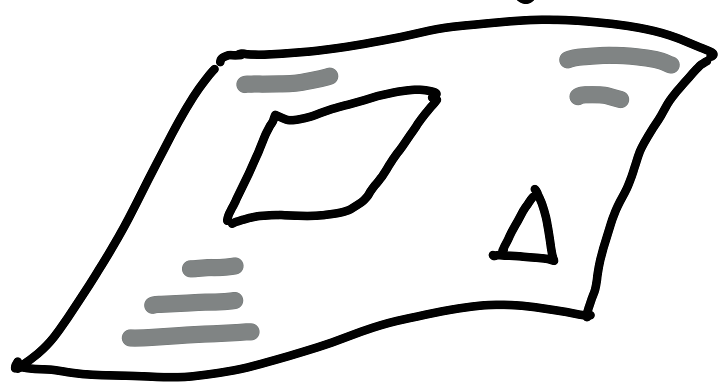
Constraint matrix is TU $\iff$ $[\partial_{p+1}]$ is TU.

ORIENTABLE MANIFOLDS

Consistent orientation of $(p+1)$-manifold M: Orient $(p+1)$-simplices s.t. $(p+1)$-boundary is carried by ∂M.

possibly empty





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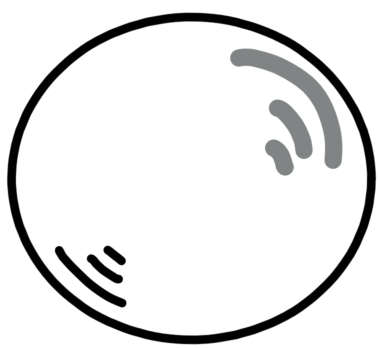

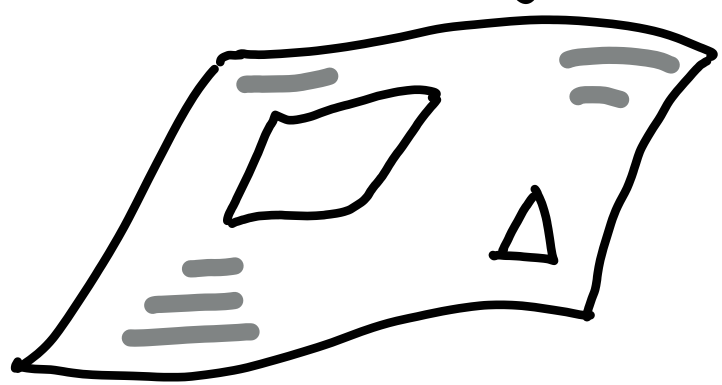

Theorem 1. For a finite simplicial complex triangulating a compact orientable manifold, $[\partial_{p+1}]$ is TU.

ORIENTABLE MANIFOLDS

Proof. Each p-simplex τ is a face of one or two $(p+1)$-simplices. So, the row of $[\partial_{p+1}]$ for τ has at most two non-zeros, and if there are two, they are $+1$ and -1.

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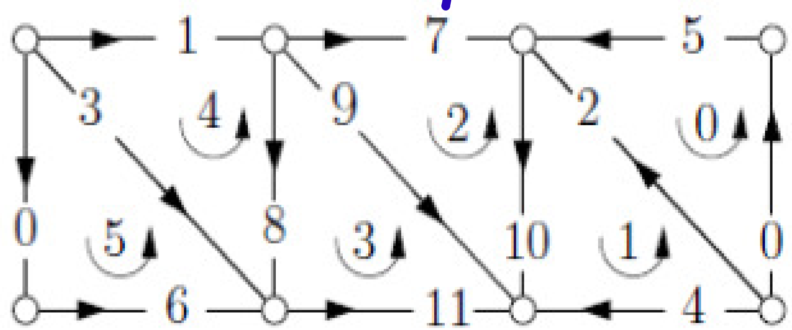
Also observed by John Sullivan (1992)

Non-Orientable Manifolds

Möbius strip:



$[\partial_2]$ for Möbius strip :

$$
\begin{array}{r}
\\ 0: \\ 1: \\ 2: \\ 3: \\ 4: \\ 5: \\ 6: \\ 7: \\ 8: \\ 9: \\ 10: \\ 11:
\end{array}
\begin{bmatrix}
0: & 1: & 2: & 3: & 4: & 5: \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
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\end{bmatrix}
$$

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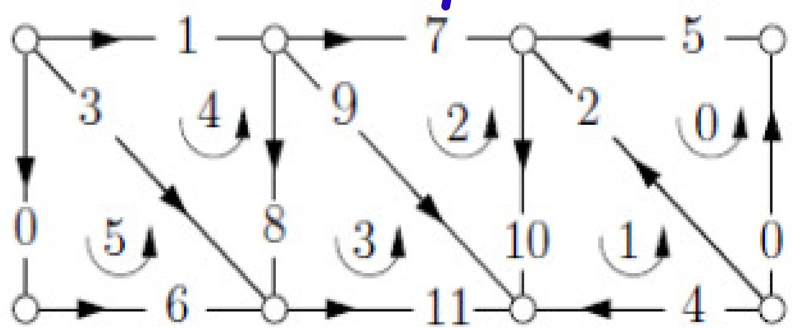

$$S = \begin{array}{c} \begin{array}{cccccc} 5 & 4 & 3 & 2 & 1 & 0 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \end{array} \begin{array}{c} \\ 0 \\ 3 \\ 8 \\ 9 \\ 10 \\ 2 \end{array}$$

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$\det S = -2. \Rightarrow [\partial_2]$ is not TU.

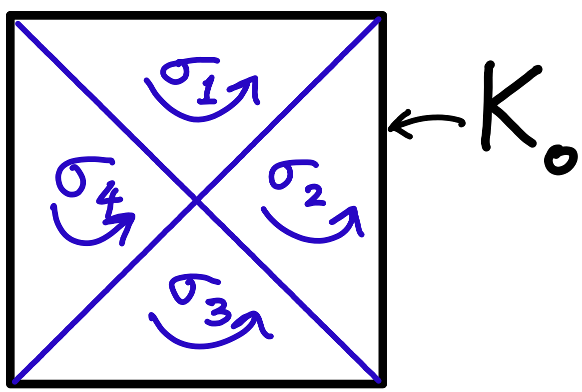
RELATIVE HOMOLOGY

K_0: subcomplex of K.

$$C_p(K, K_0) = C_p(K)/C_p(K_0)$$

is group of relative chains of K modulo K_0.

K



$\sum_{i=1}^{4} n_i \sigma_i$ is a relative 2-cycle of K modulo K_0 iff

$$n_1 = n_2 = n_3 = n_4.$$

($\sum_{i=1}^{4} n_i \sigma_i$ is a 2-chain, but NOT a 2-cycle of K)

Relative Boundary

$$\partial_p(K,K_0) : C_p(K,K_0) \to C_{p-1}(K,K_0)$$

induced by $\partial_p : C_p(K) \to C_{p-1}(K)$

$Z_p(K,K_0) = \ker \partial_p$ relative cycles

$B_p(K,K_0) = \operatorname{im} \partial_{p+1}$ relative boundaries

$$H_p(K,K_0) = Z_p(K,K_0)/B_p(K,K_0)$$

Relative Boundary Matrix

$$\partial_{p+1}(K, K_0) : C_{p+1}(K, K_0) \longrightarrow C_p(K, K_0)$$

From original $[\partial_{p+1}]$,

* include columns corresponding to $(p+1)$-simplices in K; and from this submatrix,
* exclude rows corresponding to p-simplices in K_0.

RELATIVE BOUNDARY MATRIX

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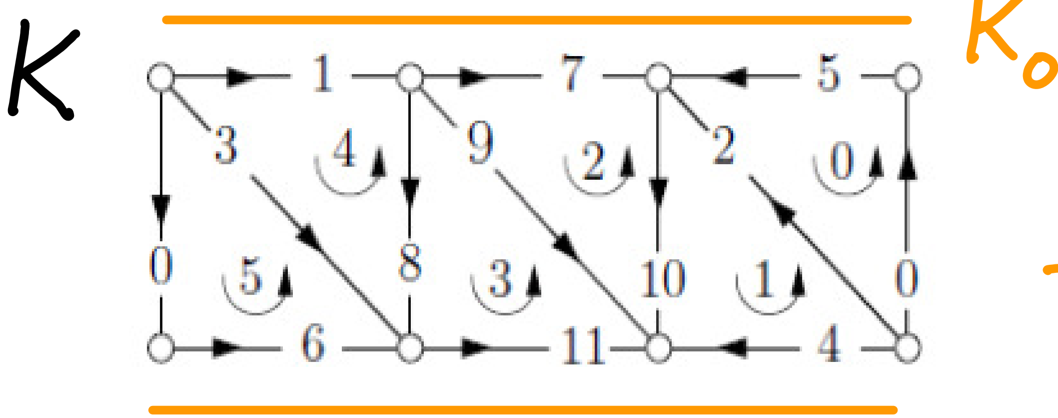
* include columns corresponding to $(p+1)$-simplices in K; and from this submatrix,
* exclude rows corresponding to p-simplices in K_0.

$K \leftarrow L$, $K_0 \leftarrow L_0$, with $L_0 \subset L \subset K$

$[\partial_{pt1}(K,K_0)]$ for Möbius Strip



$$S = \begin{array}{c} \\ \end{array}\begin{array}{cc} \begin{array}{cccccc} 5 & 4 & 3 & 2 & 1 & 0 \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} & \begin{array}{c} 0 \\ 3 \\ 8 \\ 9 \\ 10 \\ 2 \end{array} \end{array}$$

$[\partial_2]$ for Möbius strip :

$$\left[\begin{array}{c|cccccc} & 0: & 1: & 2: & 3: & 4: & 5: \\ 0: & 1 & 0 & 0 & 0 & 0 & 1 \\ 1: & 0 & 0 & 0 & 0 & -1 & 0 \\ 2: & -1 & 1 & 0 & 0 & 0 & 0 \\ 3: & 0 & 0 & 0 & 0 & 1 & -1 \\ 4: & 0 & -1 & 0 & 0 & 0 & 0 \\ 5: & 1 & 0 & 0 & 0 & 0 & 0 \\ 6: & 0 & 0 & 0 & 0 & 0 & 1 \\ 7: & 0 & 0 & -1 & 0 & 0 & 0 \\ 8: & 0 & 0 & 0 & 1 & -1 & 0 \\ 9: & 0 & 0 & 1 & -1 & 0 & 0 \\ 10: & 0 & 1 & -1 & 0 & 0 & 0 \\ 11: & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

MAIN RESULT

Theorem 2: $[\partial_{p+1}]$ is TU iff $H_p(L, L_0)$ is torsion-free for all **pure** subcomplexes L, L_0 of K of dimensions $(p+1)$ and p, respectively, where $L_0 \subset L$.

MAIN RESULT

Theorem 2: $[\partial_{p+1}]$ is TU iff $H_p(L, L_0)$ is torsion-free for all pure subcomplexes L, L_0 of K of dimensions $(p+1)$ and p, respectively, where $L_0 \subset L$.

Proof: Uses connections of torsion coefficients of abelian groups and Smith Normal Form (SNF) of $[\partial_{p+1}]$.

TORSION

$C_p(K)$, $Z_p(K)$, $H_p(K)$: finitely generated abelian groups

Fundamental theorem of fin. gen. abelian groups:

$G = H \oplus T$ where $H \cong (\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\beta})$,

and $T \cong (\mathbb{Z}/t_1 \oplus \cdots \oplus \mathbb{Z}/t_k)$ s.t.

$t_i > 1$ and $t_i \mid t_{i+1}$ (integers)

T: torsion of G. If $T = 0$, G is torsion-free.

(ref: Munkres - Algebraic Topology)

SMITH NORMAL FORM (SNF)

Result: G, G' are free abelian groups of ranks n & m, resp.; let $f: G \to G'$ be a homomorphism. $\exists$ bases for G, G' s.t. relative to these bases, the matrix of f has the form

$$B = \left[\begin{array}{c|c} \begin{matrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_\ell \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right],$$

where $b_i \geq 1$ and $b_1 | b_2 | \cdots | b_\ell$.

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where $b_i \geq 1$ and $b_1 | b_2 | \cdots | b_\ell$.

For $[\partial_{pH}]$, if $b_i > 1$ for some i, K has torsion.

SNF AND TU OF $[\partial_{p+1}]$

If K has torsion, then in $\mathrm{SNF}([\partial_{p+1}])$, $b_i > 1$ for some $1 \le i \le l$.

$$\Rightarrow b_1 \cdot b_2 \cdots b_i > 1.$$

Result (Smith, 1861): $b_1 \cdot b_2 \cdots b_i$ is the gcd of all $i \times i$ determinants of $[\partial_{p+1}]$.

(quoted in Schrijver, 1986)

$\Rightarrow [\partial_{p+1}]$ is not TU.

Testing Relative Torsion in K

Questions:

- → Is $H_p(L, L_0)$ torsion-free for ALL subcomplexes L, L_0 of K with $L_0 \subset L$?
- → Does $H_p(L, L_0)$ have torsion for SOME subcomplexes L, L_0 of K with $L_0 \subset L$?

TESTING RELATIVE TORSION IN K

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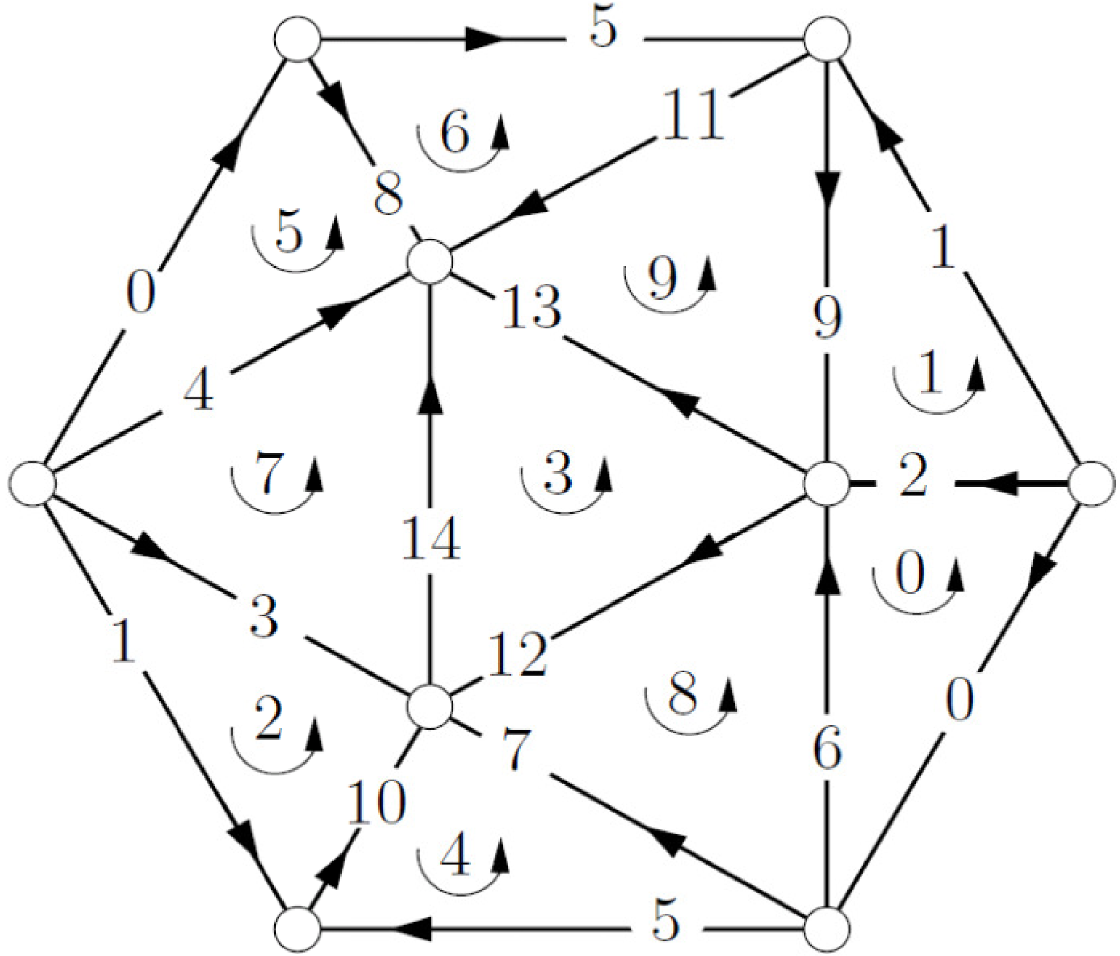
Can answer in polynomial time - check if $[\partial_p]$ is TU.

Seymour (1980) - Decomposition of regular matroids

Projective Plane



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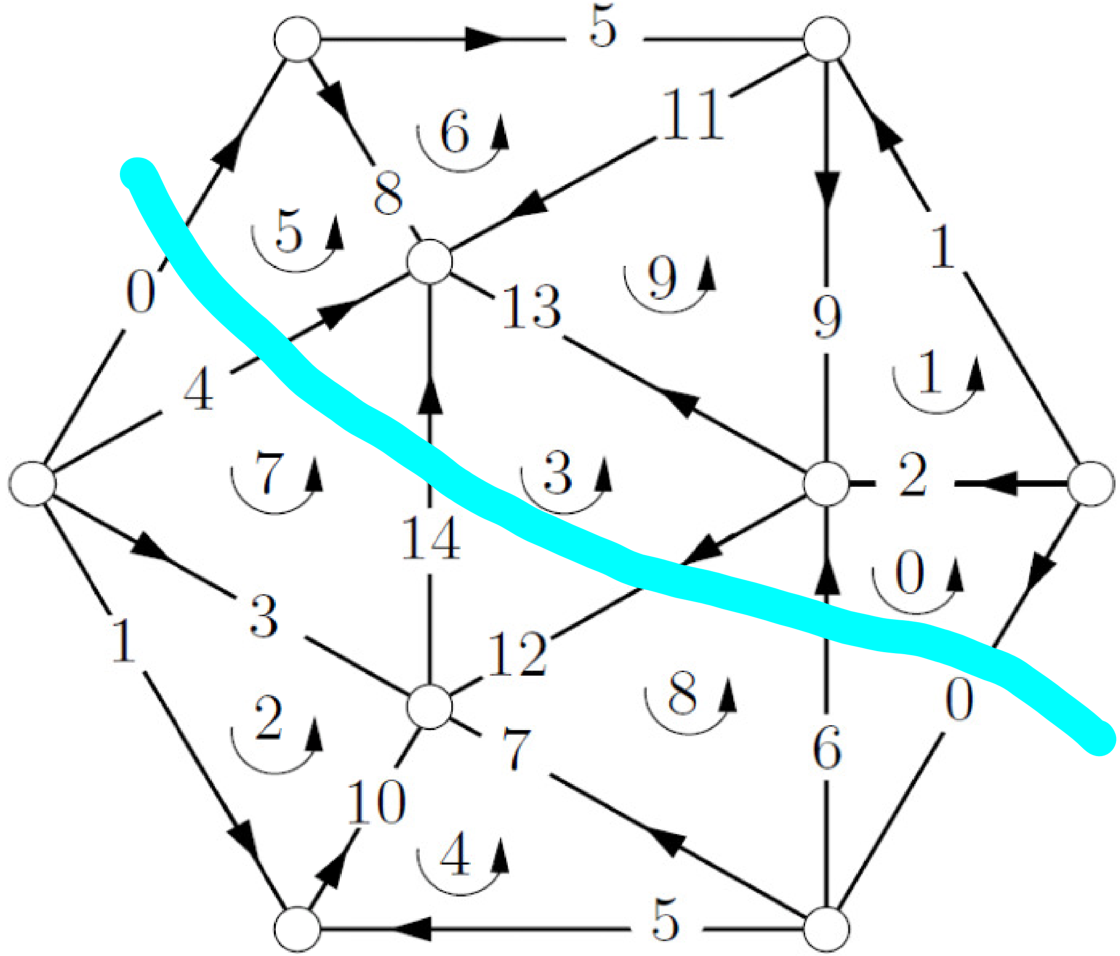

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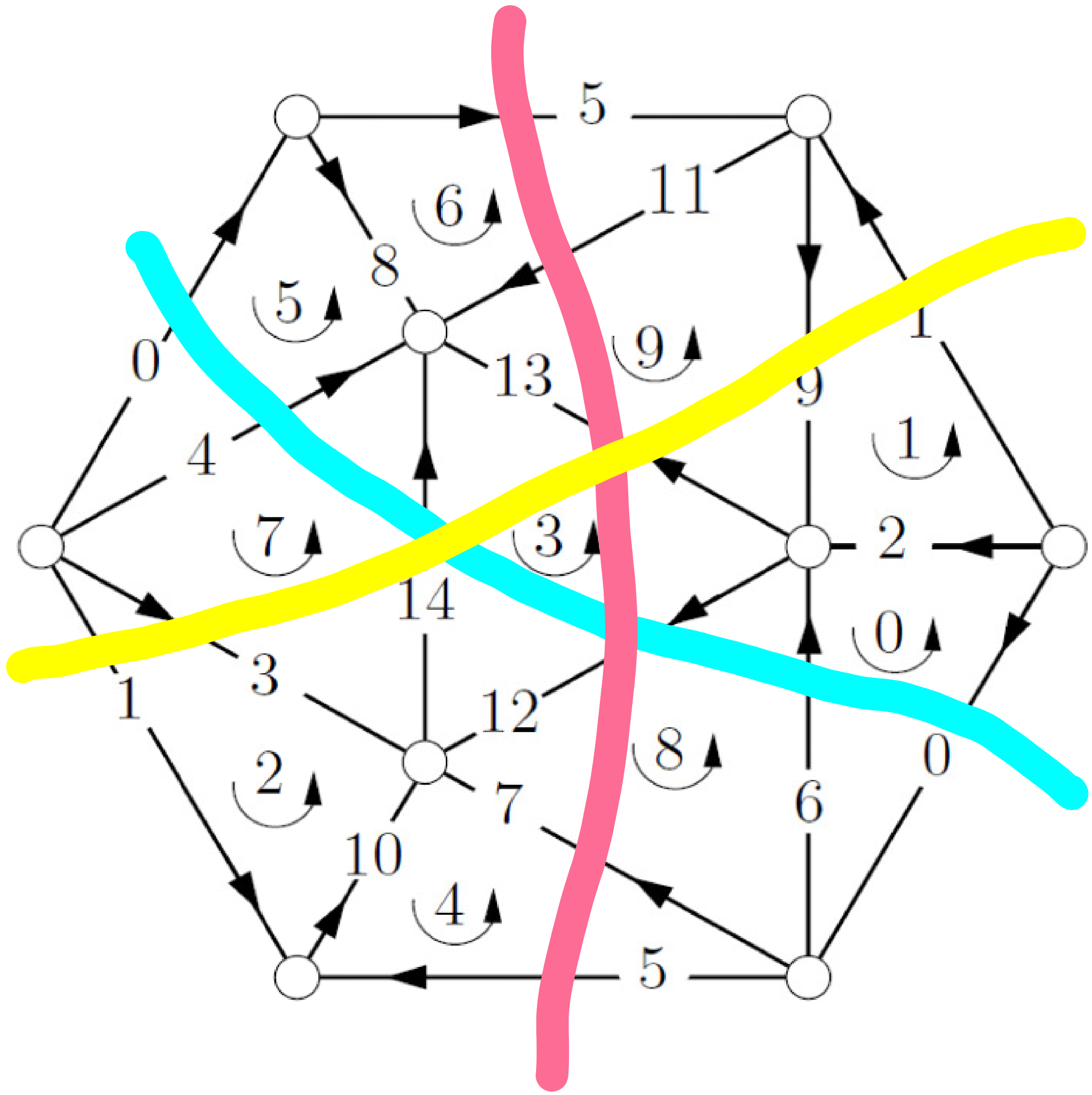

Möbius Cycle Matrices

$$C = \begin{bmatrix} 1 & 0 & & & \alpha \\ 1 & 1 & & \bigcirc & \\ 0 & 1 & 1 & & \\ & \bigcirc & & \ddots & \\ & & & 1 & 1 \end{bmatrix}_{n \times n}$$

$\det C = 2.$

if $\alpha = (-1)^{n+1}$, C is the boundary matrix of a Möbius band modulo its edge, up to row/column scalings by -1, and interchanges.

Möbius Cycle Matrices

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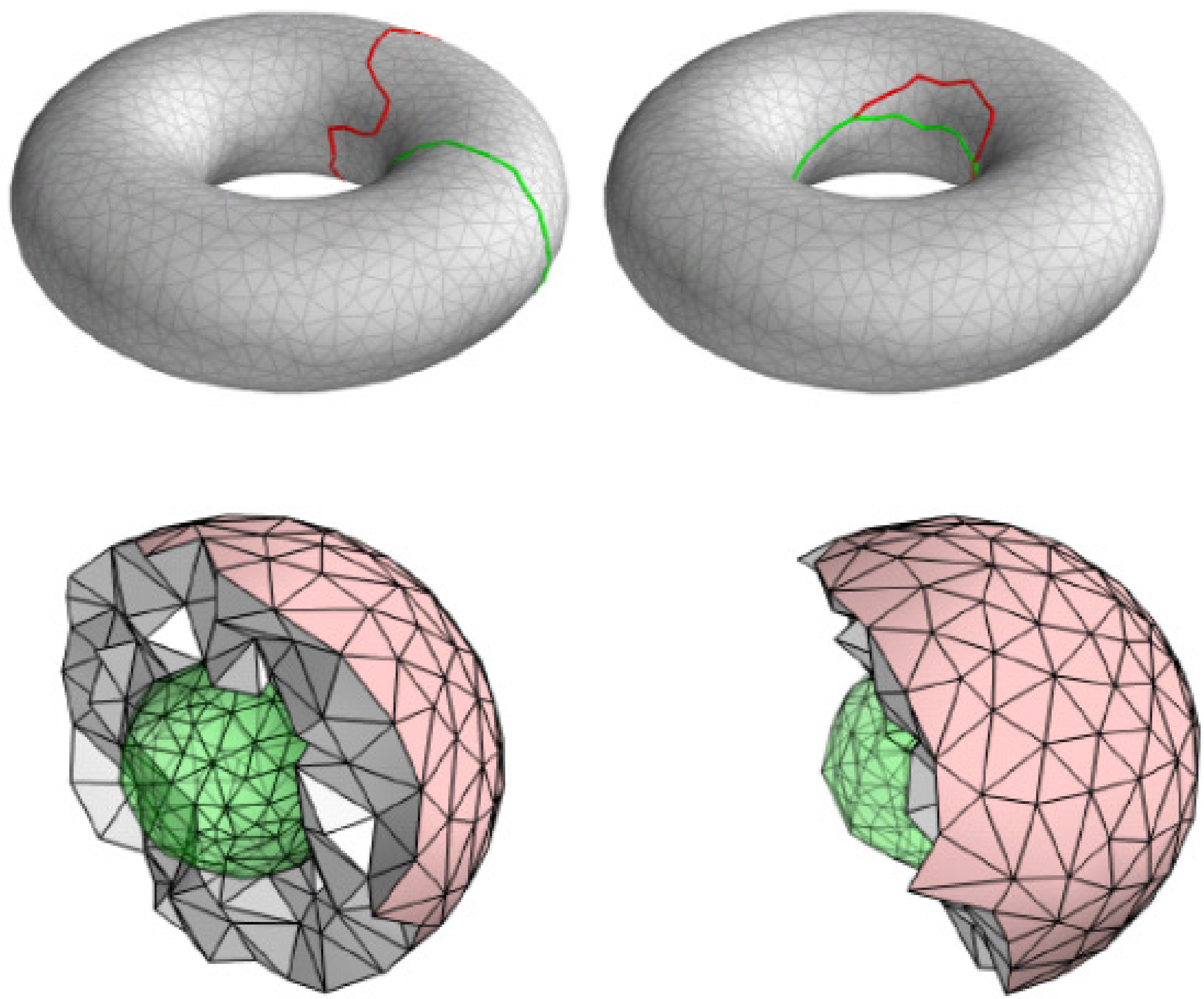
if $\alpha = (-1)^{n+1}$, C is the boundary matrix of a Möbius band modulo its edge, up to row/column scalings by -1, and interchanges.

$\det C = 2$.

If $[\partial_{p+1}]$ has such a submatrix, it is not TU.

$[\partial_{p+1}]$ has no MCMs $\overset{?}{\Longrightarrow}$ $[\partial_{p+1}]$ is TU.

Experiments



OPEN QUESTIONS

* General W, in place of $W = \mathrm{diag}[w_1, \dots, w_m]$?
* Can we still get integral solution in the presence of relative torsion?
* Faster algos to solve the OHCP LP?
* LP for optimal homology basis?